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ELEMENTS

OF THE

CONIC SECTIONS,

BY

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IN THE

University of Glasgow.

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ELEMENTS

OF THE

CONIC SECTIONS.

Book First.

OF THE PARABOLA.

DEFINITIONS.

BOOK I.
Fig. 1

I. A STRAIGHT line AB, and C a point without it, are given in position. On the plane of ABC, there is placed a ruler DEF, having its side DE applied to AB, and its other side EF on the same side of AB with the point C. A string FGC is taken equal in length to EF: and one end of this string being fixed in F, and the other in C, a part of it FG is, by means of a pin G, brought close to the side FE of the ruler; then, the string being kept uniformly tense by the pin, the side DE of the ruler is moved along AB: and thus the point of the pin, as it moves onwards with the ruler, describes upon the plane a line named the PARABOLA. This line may be extended to a distance from the point C, exceeding any given distance, provided the length of the side FE of the ruler employed, be greater than that given distance.

II. The straight line AB is named the *directrix.*

III. And the point C is named the *focus* of the parabola.

IV. A straight line perpendicular to the directrix, is named a *diameter;* and the point where a diameter meets the parabola, is named the *vertex of that diameter*; and the diameter which passes through the focus, is named the *axis of the parabola;* and the vertex of the axis is named the *principal vertex.*

V. When a straight line terminated both ways by a parabola, is bisected by a diameter, it is said to be *ordinately applied* to that diameter; or, it is named, simply, an *ordinate* to that diameter.

VI. A straight line quadruple of that segment of a diameter which is intercepted between its vertex and the directrix, is named the *latus rectum*, or the *parameter*, of that diameter.

VII. A straight line meeting a parabola only in one point, and which, when produced both ways, falls without the parabola, is said to *touch* the parabola in that point.

PROPOSITION I. THEOREM.

A straight line drawn perpendicular to the directrix from any point of a parabola, is equal to the straight line drawn to the focus from that same point.

Let G be a point in the parabola, and GE a straight line perpendicular to the Fig. 1.

directrix AB; draw GC to the focus C, and let EF be equal to the length of that side of the ruler which is on the same side of AB with the focus C: therefore EF is equal to the length of the string FGC: take away the common part FG, and the remainder EG will be equal to the remainder GC.

Corollary. Hence that segment of the axis which is intercepted between the focus and the directrix, is bisected in the vertex of the axis. Thus CB is bisected in H.

PROP. II. Theor.

If the distance of any point from the focus of a parabola be equal to the perpendicular drawn from the same point to the directrix, that point is in the parabola.

Fig. 1. n. 2. Let there be a parabola, the directrix of which is AB, and the focus C; and let D be a point, the distance of which from the focus is the straight line DC; from D draw DE perpendicular to the directrix. If DC be equal to DE, the point D is in the parabola.

From the center C, at the distance CD, describe a circle, meeting the axis in the point F: let H be the vertex of the axis, and join CE: then, because any two sides of a triangle are together greater (20. 1. Elements of Euclid) than the third side, CD, DE are together greater than CE; much more, then, are they together (19. 1. Elem.) greater than CB: but CD is equal to DE, as also CH (cor. 1. 1.) to HB: therefore, CD, that is, CF, is greater than CH: the parabola, therefore, with respect to its vertex H, is within the circle GDF: of consequence, it must meet the circle somewhere, since it may be extended (def. 1.) to a distance from the focus C which shall exceed any given distance. Now it meets the circle in the point D; for, if this is not true, it must meet the circle in some other point. Suppose it meet in the point L, which is on the same side of the axis with the point D, then having joined CL, draw LM perpendicular to the directrix, and LN parallel to it; and let LN meet DE in

N; and, because the point L is in the parabola, CL is (1. 1.) equal to LM; and, according to the hypothesis, CD is equal to DE; and, C being the center of the circle, CL is equal to CD: therefore LM, that is, NE, is equal to DE; which is impossible: the parabola, therefore, will not meet the circle in the point L, nor any where but in D: therefore D is a point in the parabola.

PROP. III. THEOR.

Any straight line drawn through the focus meets the parabola; and a straight line drawn from any point within a parabola to the focus, is less than the perpendicular drawn from that point to the directrix. A straight line, on the other hand, drawn from any point without a parabola to the focus, is greater than the perpendicular drawn from that point to the directrix.

Fig. 1. Let there be a parabola, the directrix of

which is AB, and the point C the focus; any straight line drawn through C meets that parabola.

First, if CB, a straight line drawn through the focus, be perpendicular to the directrix, the point H, bisecting (cor. 1. 1.) the segment, intercepted between the focus and the directrix, is in the (2. 1.) parabola: but if any other straight line CP be drawn through the focus, bisect the angle BCP by the straight line CM, and let CM meet the directrix in M, and draw MN parallel to the axis BC: then, because the angles PCM, CMN are together less than two right angles, for each of them is less than one right angle, the straight lines CP, MN meet each other; let them meet in the point O, then the angle OCM is equal to the angle CMO; for each of the two is equal to (29. 1. Elem.) BCM; therefore, OM is (6. 1. Elem.) equal to OC: consequently, the point O is in (2. 1.) the parabola.

In proceeding to demonstrate the other part of the proposition: First, let there be any point K within the parabola, that is, let it be on the same side of it with the focus C, BOOK I. and draw KL at right angles to the directrix; draw likewise KC to the focus; KC is less than KL. Let CK meet the parabola in O, and let there be drawn to the directrix the straight line OM parallel to KL, and let OL be joined. Since, then, the point O is in the parabola, OC is equal to OM; but OM is (19. 1. Elem.) less than OL; much more therefore is (OM, or) OC less than (20. 1. Elem.) OK and KL together: take away the common part OK, and the remainder KC is less than the remainder KL.

Next, let the point Q be without the parabola, and draw QR at right angles to the directrix; QC drawn to the focus is greater than QR. Let QC meet the parabola in O, and draw OM parallel to QR, and join QM; therefore, because CO is equal to OM, CQ is equal to MO together with OQ; but MO together with OQ, is greater than QM; much more, then, are they greater than QR. QC is, therefore, greater than QR.

Cor. Hence it is evident, that any point is within or without a parabola, according as the distance of that point from the focus is less or greater than a perpendicular drawn from that same point to the directrix.

PROP. IV. Theor.

A perpendicular to the directrix meets the parabola only in one point; and when produced downwards, it falls within the parabola.

Let MT be perpendicular to the directrix AB; draw MC to the focus; let CO be drawn, making the angle MCO equal to CMO, and meeting MT in O; OM is consequently equal to OC; and therefore the point O is in (2. 1.) the parabola. Fig. 1.

Next, take any point T in MO produced, and join TC: since, then, the angle MCT is greater than MCO, that is, than the angle CMT, TM is greater than TC: the point T therefore is within (cor. 3. 1.) the parabola. In like manner it may be demonstrated, that any point above O in the straight line MT is without the parabola.

PROP. V. Theor.

If from a point in a parabola a straight line be drawn to the focus, and if from the same point a straight line be drawn perpendicular to the directrix; the straight line which bisects the angle contained by these two straight lines, touches the parabola in the said point. Also, a straight line drawn through the vertex of the axis at right angles to the axis, touches the parabola.

Fig. 2.

1. D being a point in a parabola, and DC drawn from D to the focus, and DA drawn perpendicular to the directrix AB; DE that bisects the angle CDA, touches the parabola in the point D.

In DE take any other point F; and having joined FA, FC, AC, draw FG perpendicular to the directrix; then, because DA is equal (1. 1.) to DC, DF common, and the angle FDA equal to FDC; FC is equal (4. 1. Elem.) to FA; and, consequently, greater than FG; therefore the point F is without the (cor. 3. 1.) parabola; and, consequently, the straight line DE touches the parabola (def. 7.) in the point D.

Fig. 2.

2. HK, a straight line drawn through the vertex of the axis, at right angles to the axis, touches the parabola. In HK take any point K; from which draw KL perpendicular to the directrix; and join KC: and, because KC is greater (19. 1. Elem.) than CH, that is, than (cor. 1. 1.) HB, that is, than KL, KC is greater than KL; therefore the point K is without the parabola, and HK touches the parabola.

Cor. 1. This proposition points out a method of drawing a straight line that will touch a parabola in a given point, provided the directrix and the focus be given in position.

Cor. 2. And since it has been proved, that all straight lines that touch a parabola, fall without it towards the same parts, that curve, it is plain, is every where convex on the side on which the touching lines are, but concave on the contrary side.

PROP. VI. PROBLEM.

The directrix and the focus of a parabola, and a straight line not parallel to any diameter, being given in position; to draw a straight line parallel to the straight line given in position, which will touch the parabola.

Fig. 2. AB being the directrix and C the focus of a parabola, and MN a straight line not parallel to any diameter; it is required to draw a straight line parallel to MN, which will touch the parabola.

From the focus C draw CO perpendicular to MN, and meeting the directrix A; and having bisected AC in E, draw ED parallel to MN, and meeting the diameter through A in the point D, and join CD; then, in the triangles ADE, CDE, AE is equal to CE, ED common, and the angles at E right angles; DA, therefore, is equal to DC; and, consequently, the point D is in the (2. 1.) parabola: and, since the angle ADE is equal to the angle CDE, the straight line DE, as was shown in the preceding proposition, touches the parabola.

PROP. VII. THEOR.

Fig. 3. 4. n. 1. 2.

If from a point E in a parabola, there be drawn a straight line EG, neither parallel to the axis, nor bisecting the angle contained by the diameter passing through that point, and a straight line drawn from the same point to the focus; the straight line EG cuts the parabola in one other point, but not in more than one.

From the focus C let a perpendicular be drawn to EG, and let it meet the directrix in A; and, making A*f* equal to AF, through *f* draw *fe* parallel to the diameter FE, and let *fe* meet EG in *e*; the point *e* will be in the parabola.

Fig. 3.

There are two cases.—First, where EG passes through the focus. Because EC is equal to EF, and each of the angles ECA, EFA a right angle; therefore AC is (5. and 6. 1. Elem.) equal to AF; and, consequently, it is equal to A*f*; and each of the angles AC*e*, A*fe* is a right angle: *e*C is, therefore, equal to *ef*; and therefore the point *e* is in the (2. 1.) parabola.

Fig. 4. n. 1. 2. In the second, EG does not pass through the focus. From the centre E, at the distance EC, describe a circle, meeting CA again in H; and describe another circle through the points C, H, *f*; then, because EC is equal to EF, and that EFA is a right angle, the circle described from the centre E touches (16. 3. Elem.) the directrix in F: therefore the rectangle CAH is equal to (36. 3. Elem.) the square of AF, that is, to the square of A*f*: therefore A*f* touches the circle (37. 3. Elem.) *f*CH; and the centre of this circle is (19. 3. Elem.) in *fe*; it is also in GE, which bisects CH at right angles: it is, therefore, in the point *e* where *fe*, GE intersect each other. *e*C, therefore, is equal to *ef*; and, therefore, the point *e* is in the (2. 1.) parabola.

It is evident, that EG cuts the parabola no where but in the points E, *e* : for, if possible, let EG cut it also in another point ε; and let εφ be drawn perpendicular to the directrix AB; a circle, thus described from the centre ε, distance εC, passes through H, and touches the directrix in the point φ, at a distance from the point A, less or greater than that of the point F or *f* from A; and the square of φA being equal to the rectangle CAH, is equal to the square of FA: which is absurd.

Fig. 4. n. 1. 2.

n. 1. 2.

Cor. Of all the straight lines that can be drawn from any point of a parabola, only one can touch the parabola; for the diameter through the point falls (4. 1.) within the parabola; and any other straight line, except that which bisects the angle contained by the diameter through the point, and the straight line drawn from the point to the focus, meets the parabola again in another point.

PROP. VIII. Theor.

If from the focus C of a parabola, a perpendicular CG be drawn to any straight

Fig. 4. n. 1. 2.

line LG, meeting the directrix in A; if the segment of the perpendicular intercepted between the focus and the straight line, is not greater than its other segment intercepted between the straight line and the directrix, that is if CG be not greater than GA, the straight line LG, necessarily, meets the parabola.

When the segments CG, GA are equal, it is plain from what was demonstrated in Prop. 6. that the straight line LG touches the parabola in the point where the diameter through A meets LG.

But if CG be less than GA, take GH equal to GC; and from the point A, and on either side of A, place, in the directrix, AF, or A*f*, such, that the square of either may be equal to the rectangle CAH; and, having described a circle through the points C, H, F, draw FE perpendicular to AF, and let FE meet LG in E: and the square of AF being equal to the rectangle CAH, AF touches the circle

PLATE I.

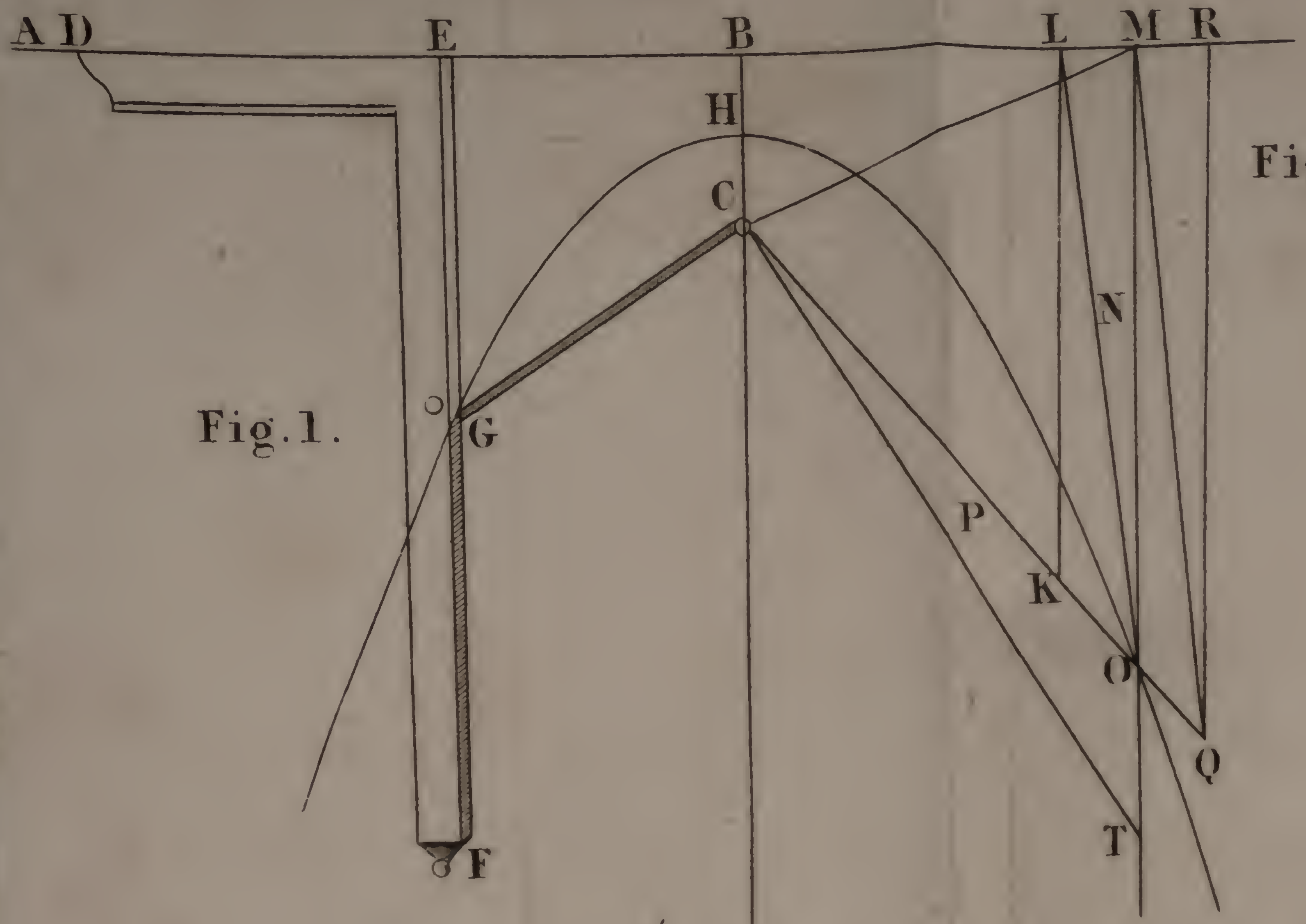


Fig. 1.

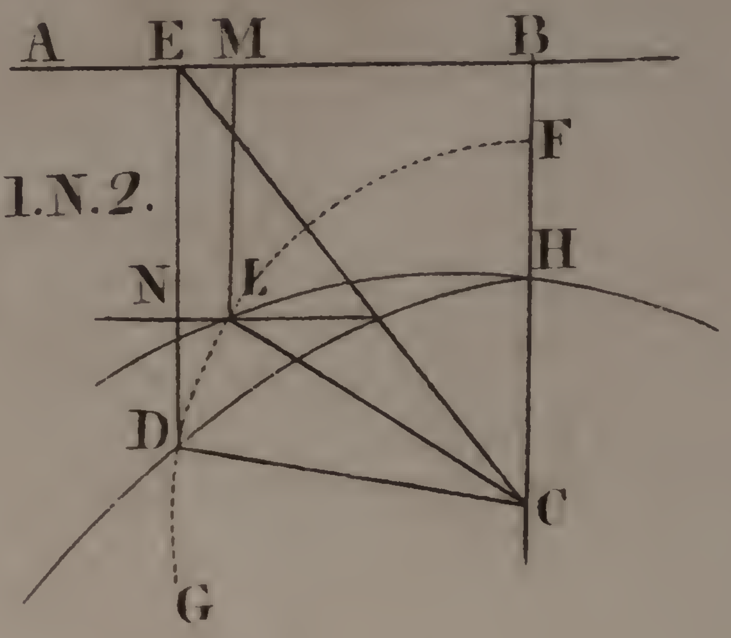
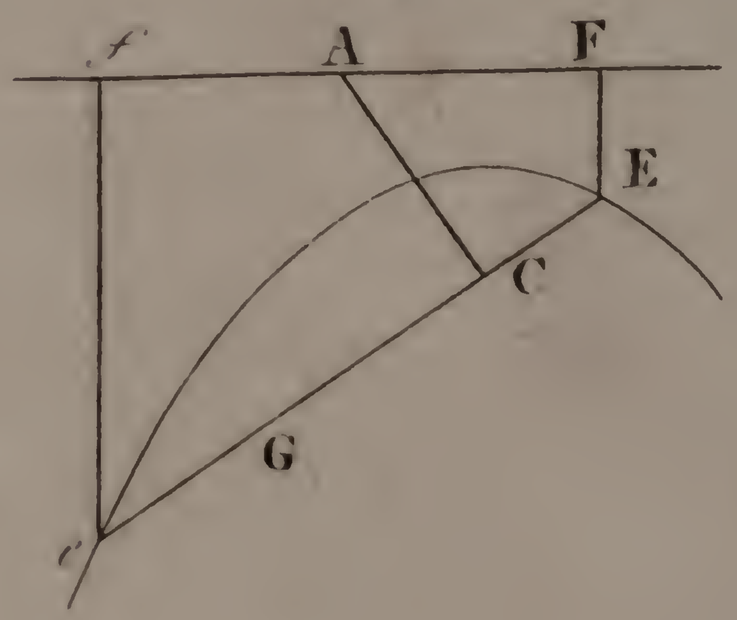
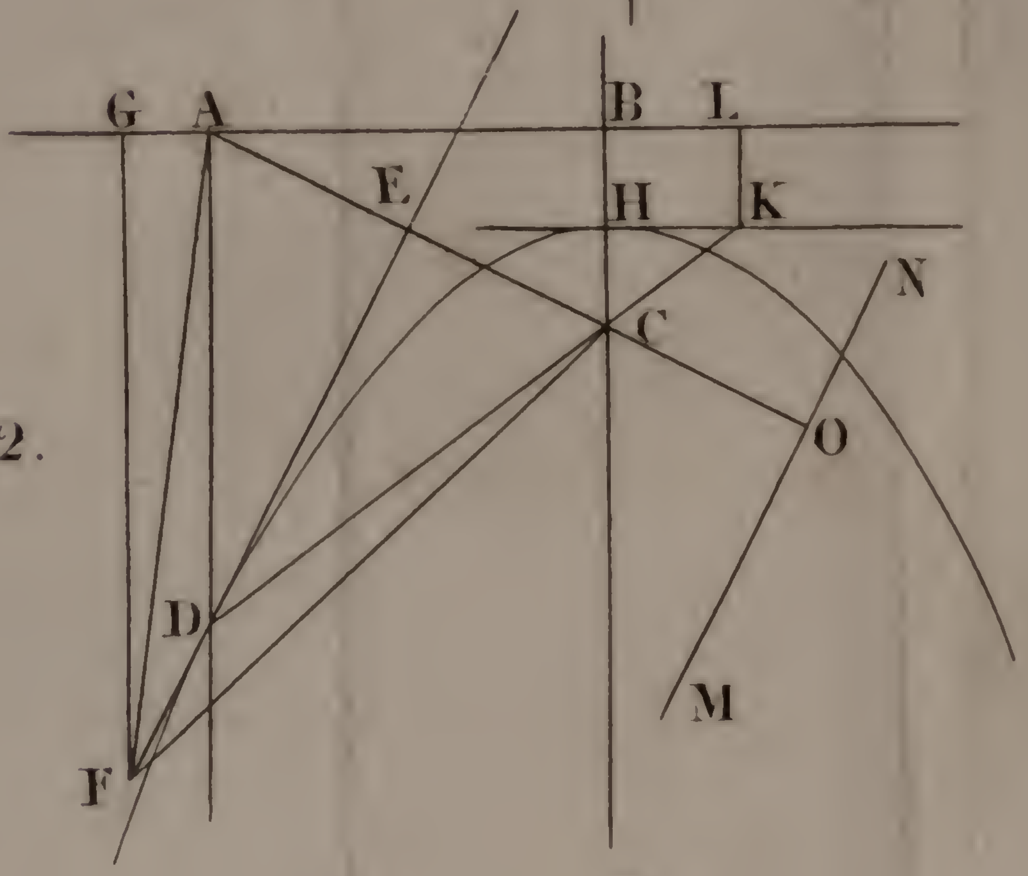


Fig. 1. N. 2.

Fig. 3.

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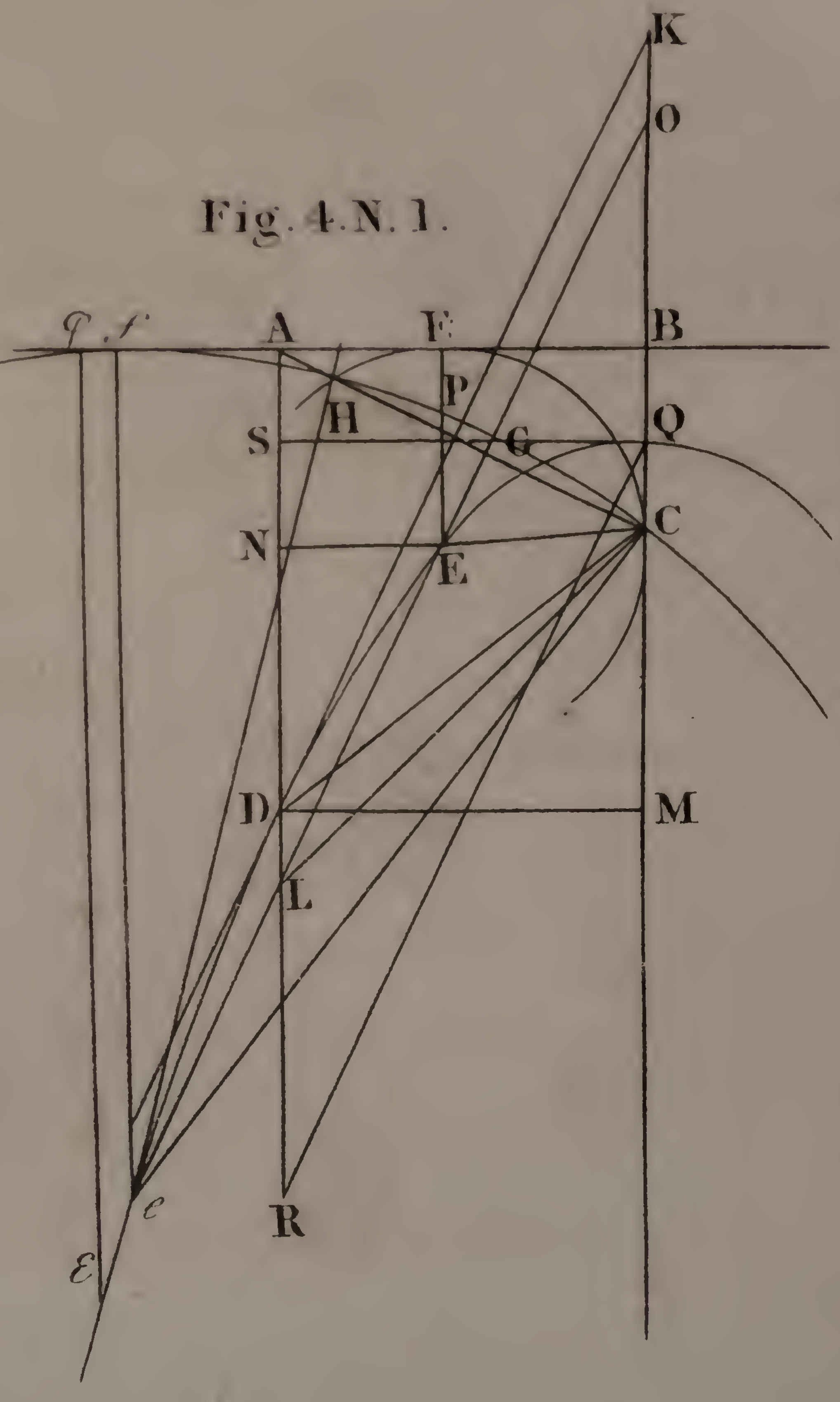
Fig. 4. N. 1.

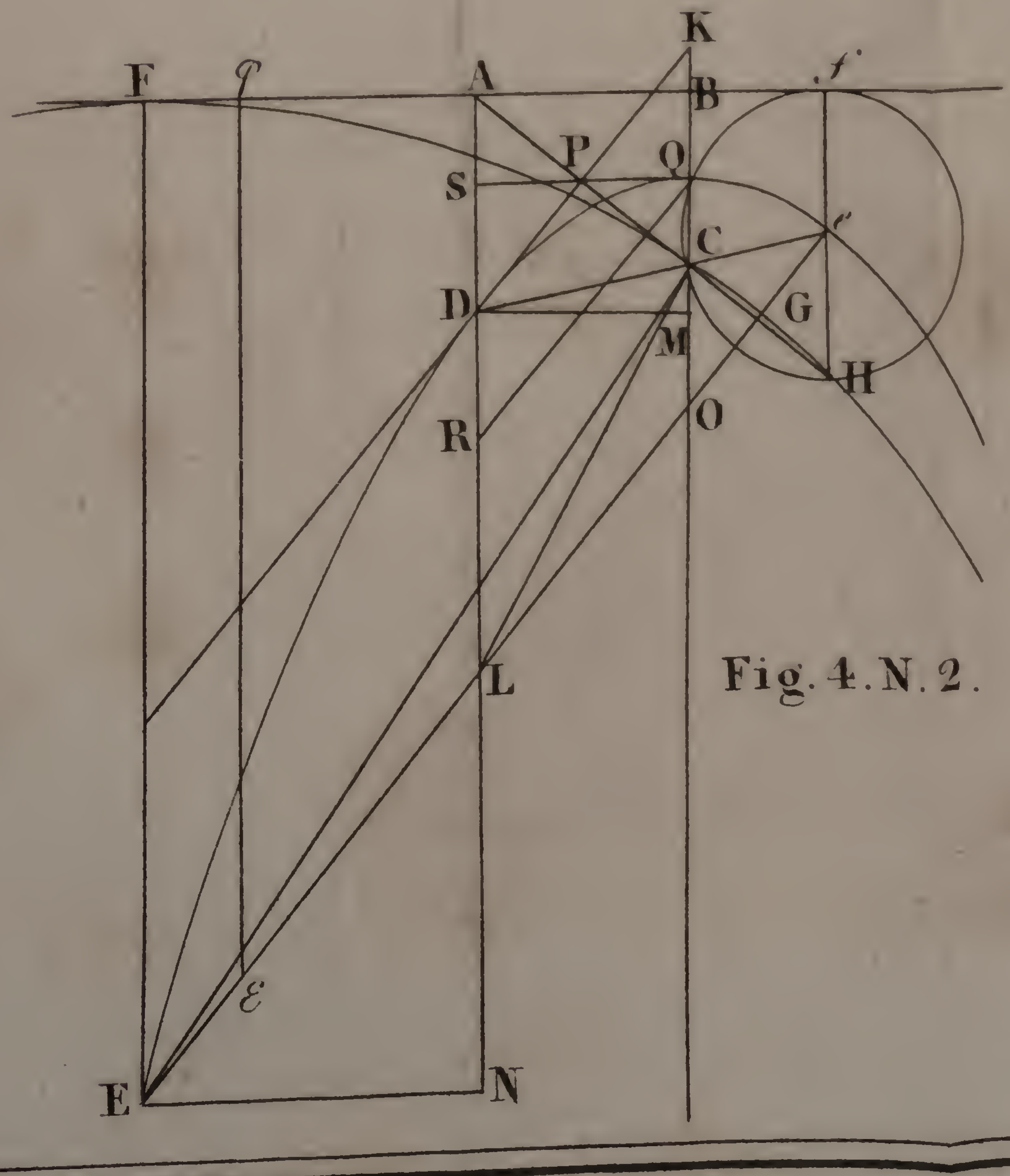
Fig. 4. N. 2.

in F; and therefore the centre of the circle is in FE: but as CH is bisected at right angles by the straight line LG, the centre of the circle is likewise in LG: it is, therefore, in E, the point where FE, LG intersect each other: hence EC is equal to EF; and, therefore, the point E is in the parabola. In like manner, if *ef* drawn perpendicular to the directrix meets LG in *e*, the point *e* is in the parabola.

Cor. Hence any straight line passing through a point within a parabola, meets the parabola.

Case 1. If the straight line is a diameter, it is evident, from Prop. 4. 1. that it meets the parabola.

Case 2. When the straight line is not a diameter. Let LG pass through the point L within the parabola; it will meet the parabola. From the focus C let the straight line CG be drawn at right angles to LG, and let it meet the directrix in A; and join LC, LA: then because the point L is within the parabola; a straight line drawn from L perpendicular to the directrix, is greater (3. 1.)

than LC: consequently, LA, which is not less than this perpendicular, is greater than LC: AG, therefore, is greater than (47. 1. Elem.) GC; and therefore LG meets the parabola.

PROP. IX.

The angle contained by a diameter of a parabola, and a straight line drawn from the vertex of that diameter to the focus, is bisected by the straight line that touches the parabola in that vertex.

Fig. 5. The angle ADC, contained by the diameter AD, and the straight line DC, is bisected by DE, a straight line touching the parabola in the vertex D: for if the angle ADC is not bisected by DE, it is possible for some other straight line to do it; and this other straight line also will touch the parabola (5. 1.): which is absurd.

Cor. 1. On the other hand, if AD be a diameter, and ED touch the parabola in the vertex D of AD, and if the angle ADE be equal to the angle CDE, DC passes through the focus: or, if DC passes through the focus, DE touching the parabola in D, and the angle ADE being equal to the angle CDE; DA is a diameter.

Cor. 2. If from any point D which is in a parabola, but which is not the vertex of the axis, a straight line DE be drawn touching the parabola, the angle EDA contained towards the directrix by the straight line DE and the diameter DA, is less than a right angle; for the angle ADC, which is the double of EDA, is less than two right angles.

PROP. X. Theor.

If from a point in a parabola a straight line be drawn touching the parabola, and if from the same point a perpendicular be drawn to the axis; the segment of the axis intercepted between the perpendicular and the line touching the parabola, is bisected in the vertex of the axis.

Fig. 5. Let D be a point of the parabola, and let a line drawn from D to E touch the parabola, and DH be perpendicular to the axis; the segment EH of the axis is bisected in F, the vertex of the axis.

Through D let DA be drawn perpendicular to the directrix; let DC be drawn to the focus, and let the axis meet the directrix in B: and because the angle CDE is equal to the angle ADE (9. 1.), that is, to the alternate angle CED, CE is equal to CD, or DA, that is, to HB; and CF is equal to FB; therefore the remainder FE is equal to the remainder FH.

PROP. XI. THEOR.

Every straight line parallel to a straight line that touches the parabola, and terminated both ways by the parabola, is bisected by the diameter passing through the point of contact, that is, it is ordinately applied to this diameter.

Fig. 4. n. 1. 2. The straight line *Ee*, which is terminated in the points E, *e*, being parallel to DK, a straight line touching the parabola; and AD, the diameter which passes through the point of contact D, meeting E*e* in L; LE is equal to L*e*.

Let AD meet the directrix AB in A; from the points E, *e* to the directrix, draw the perpendiculars EF, *ef;* and from the focus C draw CA meeting E*e* in G; from the centre E, distance EC, describe a circle meeting CA again in H; this circle will touch the directrix in F: join DC: then, because DA is equal to DC, and the angle ADK equal to (9. 1.) CDK, DK is (4. 1. Elem.) perpendicular to AC: and, therefore, E*e* is likewise at right angles to AC: and because E is the centre of the circle CFH, CG is equal to (3. 3. Elem.) GH: join *e*C and *e*H, and *e*C will be equal (4. 1. Elem.) to *e*H: a circle, therefore, described from the centre *e*, and at the distance *e*C, passes through H; and *e*C being equal to *ef*, it passes likewise through *f*: therefore, since the straight line F*f* touches the circles, and the straight line AHC cuts them, the square of AF is equal to the (36. 3. Elem.) rectangle CAH, that is, to the square of A*f*: therefore AF is equal to A*f*; but FE, AL, *fe*, are parallels: therefore LE is (2. 6. Elem.) equal to L*e*.

Cor. 1. Or, if a straight line E*e*, terminated both ways by a parabola, be bisected by the diameter AL, it is parallel to the tangent which passes through D, the vertex of AL: for if the straight line touching the parabola in the point D, be not parallel to LE, let another straight line be drawn touching the parabola, and parallel to LE; then the diameter which passes through the point where this other straight line touches the parabola, bisects the straight line E*e*: but, according to the hypothesis, the same E*e* is bisected by the diameter AL: which is absurd.

Cor. 2. All straight lines ordinately applied to any diameter, are parallel to one another.

Cor. 3. If two or more parallels be terminated both ways by a parabola, the diameter which bisects the one bisects also the other, or the rest of them: for the one that is bisected by a diameter, is parallel to the straight line touching a parabola in the vertex of that diameter; and consequently the other, or the others, is, or are, parallel to the same straight line that touches the parabola in that vertex; and, therefore, is, or are, bisected by the same diameter.

Cor. 4. Any straight line, on the contrary, which bisects two parallels terminated both ways by a parabola, is a diameter: for if it is not, it is possible for some other straight line bisecting one of the parallels to be a diameter; and being a diameter, this other straight line must also bisect the other of them: but, according to the hypothesis, the former of the straight lines bisects both the parallels: which is absurd. And if from the point of contact a straight line be drawn bisecting another straight line parallel to the tangent, and terminated both ways by the parabola, that straight line is a diameter: for if it be not, let a diameter be drawn through the point of contact, this diameter must also bisect the parallel to the tangent: which is absurd.

Cor. 5. And a straight line drawn through the vertex of a diameter, so as to be parallel to straight lines ordinately applied to that diameter, touches the parabola. This is manifest from cor. 1.

PROP. XII. THEOR.

If from a point of a parabola a straight line be drawn perpendicular to a diameter, and if from the same point a straight line be ordinately applied to that diameter; the square of the perpendicular is equal to the rectangle contained by the abscissa of the diameter and the *latus rectum* of the axis.

(*N. B.* An *abscissa* is the segment intercepted betwixt the vertex of a diameter and a straight line ordinately applied to that diameter.)

Case 1. When the diameter is the axis of the parabola.

Fig. 5. Let D be a point in a parabola, and DH a perpendicular to the axis BC; DH will be parallel to (5. 1.) the straight line touching

the parabola in the vertex of the axis; and therefore will be ordinately (11. 1.) applied to the axis: draw DC to the focus, and DA perpendicular to the directrix AB, and let F be the vertex of the axis; then, because HB is equal to DA, that is, to DC, the square of HB is equal to the square of DC, that is, to the square of DH, together with the square of HC: but, since BF is equal to FC, the same square of HB is equal to four times the rectangle HFC, together with the (8. 2. Elem.) square of HC: therefore the square of DH, together with the square of HC, is equal to four times the rectangle HFB, together with the square of HC: Therefore the square of HD is equal to four times the rectangle HFB, that is, to the rectangle contained by the abscissa HF, and the parameter of the axis.

Case 2. When the diameter to which the perpendicular is drawn is not the axis.

Fig. 4. n. 1. 2.

Let EN be perpendicular to the diameter AD; let EL be an ordinate to AD, and D the vertex of the same AD; the square of EN is equal to the rectangle contained by the abscissa LD and the parameter of the axis.

Draw DK parallel to LE; DK will therefore (5 cor. 11. 1.) touch the parabola in D: and let DK meet the axis in K; let EF be drawn at right angles to the directrix; and let a circle be described from the centre E, distance EF; and this circle will touch (cor. 16. 3. Elem.) the directrix in F, and pass through the focus C: let AC be joined, and let it meet the circumference of the circle again in H, and the straight lines DK, LE in the points P, G; and let LE meet the axis in O.

Because the angles (9. of this book, 4. 1. Elem.) CPK and CBA are right angles, and the angle BCP common, the triangles CBA, CPK are equiangular: AC, therefore, is (4. 6. Elem.) to CB, as CK to CP, that is, as (2. 6.; 16. 5. Elem.) OK to GP: consequently, the rectangle, contained by CA, GP is equal to (16. 6. Elem.) that contained by OK, CB: but because CA is (9. of this book, and 4. 1. Elem.) the double of CP, and CH the double of CG, AH is double of GP;

and, consequently, the rectangle CAH is equal to twice the rectangle CA, GP, that is, to twice the rectangle OK, CB: but, the square of EN, or of AF, is equal (36. 3. Elem.) to the rectangle CAH: it is, therefore, equal to twice the rectangle OK, CB, that is, to the rectangle contained by the abscissa LD, and the parameter of the axis.

Cor. 1. Hence the squares of perpendiculars drawn from any points of a parabola to any diameters, are to one (1. 6. Elem.) another, as the abscissas intercepted between the vertices of those diameters and the ordinates drawn from those points.

Cor. 2. The squares of straight lines ordinately applied to the same diameter, are to one another, as the abscissas between those straight lines and the vertex of that diameter. Let EL, QR be ordinately applied to the diameter DN; and let EN, QS be perpendicular to the same: because the triangle ELN is equiangular to the triangle QRS, the square of EL is to that of QR, as the square of EN to that of QS, that is, by the preceding corollary, as the abscissa DL to the abscissa DR.

Cor. 3. And if from the vertices of two diameters there be drawn straight lines ordinately applied to those two diameters, that is, if the straight line drawn from the vertex of each diameter be an ordinate to the other diameter, the abscissas between those ordinates and the two vertices are equal to each other; for the perpendiculars drawn from the two vertices to the two diameters are equal.

PROP. XIII. Theor.

If from a point of a parabola a straight line be drawn ordinately applied to a diameter, the square of half that ordinate is equal to the rectangle contained by the abscissa between that same ordinate and the vertex of that diameter, and the *latus rectum* of the same diameter.

Fig. 4. n. 1. 2.

Let AB be the directrix of a parabola, and AD a diameter, to which EL, drawn from the point E of the parabola, is ordinately applied; and through the vertex D of the diameter AD, draw DK parallel to EL; DK, of consequence, will touch the parabola: draw DM perpendicular to the axis, and from Q, the vertex of the axis, draw QR ordinately applied to the diameter DL; and, consequently, parallel to EL.

Since QR is equal to DK, its square is equal to (47. 1. Elem.) the squares of DM, MK; but the square of DM is, by the first case of the foregoing proposition, equal to four times the rectangle MQB; and since MQ is equal (10. 1.) to QK, the square of MK is equal to four times the square of MQ: therefore the square of QR is equal to four times the rectangle MQB, together with four times the square of MQ, that is, to four times (3. 2. Elem.) the rectangle QMB: but MQ, or QK, is equal to DR, and MB to DA; therefore, the square of QR is equal to four times the rectangle RDA; and since QR, EL are ordinately applied to the diameter AD, the square of QR is to the square of EL as (2. cor. of the preceding proposition) RD to LD, that is, as four times the rectangle RDA to four times the rectangle LDA: but the square of QR, as hath been proved, BOOK I. is equal to four times the rectangle RDA: therefore, the square of EL is equal to four times the rectangle LDA, that is, to the rectangle contained by the abscissa LD and the parameter of the diameter AD.

It was from the property above demonstrated, that Apollonius named the curve line, which is the subject of this book, the PARABOLA.

COR. 1. If from a point E to AD, a diameter of the parabola, a straight line EL is drawn parallel to straight lines ordinately applied to the diameter AD, and meeting the same AD below its vertex D; if the square of EL is equal to the rectangle contained by the abscissa LD and the parameter of the diameter AD; the point E is in the parabola.

Fig. 4. n. 2. For since the point L is within (4. 1.) the parabola, the straight line EL necessarily meets (cor. 8. 1.) the parabola: if, therefore, EL does not meet the parabola in the point E, on the same side of the diameter with E, let it meet it, if possible, in some other point, nearer, or more remote, from the diameter, than E is: let this other point be ε; then the square of εL is equal to the rectangle contained by LD, and the parameter of the diameter, that is, according to the hypothesis, to the square of EL; which is absurd.

Cor. 2. If from two points E, Q, one of which, Q, is in the parabola, there be drawn to the diameter AD straight lines, EL, QR, parallel to straight lines ordinately applied to AD; if the squares of the parallels be to one another as the abscissas between the parallels and the vertex of the same AD; the other point E is also in the parabola.

If the diameter LD meets the directrix in A, four times AD is its *latus rectum*: then, since the square of QR is to that of EL as RD is to LD, that is, as four times the rectangle RDA to four times the rectangle LDA; and since, by the proposition, the square of QR is equal to four times the rectangle RDA; the square of EL is equal to four times the rectangle LDA; and therefore, by the preceding corollary, the point E is in the parabola.

PROP. XIV. Theor.

If a straight line be drawn from a point of a parabola so as to be ordinately applied to a diameter, and if another straight line be drawn from the same point so as to touch the parabola, and meet that diameter; the segment (of the diameter) intercepted betwixt the ordinate and the tangent is bisected in the vertex of the diameter.

Fig. 6. A being a point of a parabola; AC drawn from A, so as to be ordinately applied to the diameter BC, and AD drawn from the same point, so as to touch the parabola, and meet BC in D; the segment CD is bisected in B, the vertex of BC.

From the vertex B let BE be drawn parallel to AD; it will be ordinately (11. 1.) applied to the diameter AE, and the abscissa BC will be equal to the abscissa (3. cor. 12. 1.) AE, that is, to BD.

Cor. 1. Conversely, AC being ordinately applied to the diameter BC, if AD be drawn making BD equal to BC, AD touches the parabola.

For if AD does not touch the parabola, let AF touch it; FB then is equal to BC: which is impossible.

Cor. 2. If a straight line touches a parabola, its segment between the point of contact, and any diameter, is bisected by a straight line touching the parabola, in the vertex of that diameter.

Let AD, a straight line touching the parabola, meet the diameter CB in the point D, and the tangent GB in G; let AC be drawn parallel to BG; AC will be ordinately applied to the (11. 1.) diameter CB; and since, by the proposition, CB is equal to BD, AG is likewise equal to GD.

PROP. XV. Prob.

To find a diameter, the axis, the *latus rectum* of the axis, the focus, and the directrix of a parabola given in position.

Fig. 7. Let two parallel straight lines AB, CD be drawn; let them be terminated in the parabola, and bisected in the points F, E; join FE, and let it meet the parabola in G; GF is (4. cor. 11. 1.) a diameter.

Next, in the diameter GF, and below its vertex G, take any point H; and through that point draw KHL perpendicular to the diameter GF, and meeting the parabola in the points K, L; and through M, the middle point of KL, draw MN parallel to the diameter GF, and meeting the parabola in N; and let NO be drawn parallel to MH: then, because MN is parallel to GH, it is a diameter: but KL is ordinately applied to MN; therefore NO touches (5. cor. 11. 1.) the parabola in N. And because MNO is a right angle, MN is (2. cor. 9. 1.) the axis: and a third proportional to NM, MK is the (13. 1.) *latus rectum* of the axis: and the distance of the focus from the vertex of the axis is equal to a fourth part of the *latus rectum* of the axis; therefore the focus is given. After the same manner is the directrix found.

PROP. XVI. Prob.

The directrix and the focus of a parabola being given in position, to describe the parabola.

Fig. 8.

Let AB be the directrix, and C the focus, and with a ruler and string describe the parabola; or as many points of the parabola as may be thought necessary may be thus found; through the focus C draw CB at right angles to the directrix, and CB will be the axis: to the axis CB draw any perpendicular LG, meeting it below its vertex F, and in the same axis place CH equal to BG; CH will thus be greater than CG; and from the centre C, distance CH, describe a circle, cutting the straight line LG in D, *d*; these points are in the parabola.

For the straight line DA, drawn to the directrix, so as to be parallel to GB, is equal to GB, that is, to CH, that is, to CD; and D, therefore, is (2. 1.) in the parabola. In the same manner it may be shown, that *d* is in the parabola.

Cor. Hence, if the directrix AB of a parabola, and F, the vertex of the axis, be given in position, the parabola may be described, by drawing FB at right angles to the directrix, and making FC equal to FB; for C will be the focus (cor. 1. 1.) In like manner, if the vertex F and focus C be given; join CF, and produce it to B, so that FB may be equal to FC; a straight line drawn through B at right angles to BC, will be the directrix: and if the axis GF, and its vertex F be given in position, and its parameter FK given in magnitude, the directrix may be found by making FB equal to a fourth part of the parameter FK, and drawing BA at right angles to the same FB. In like manner the directrix may be found, if the axis, the focus, and the parameter of the axis, be given in position. In all these cases, therefore, the parabola may be described according to the proposition.

PROP. XVII. Prob.

The axis and its vertex, and a point without the axis, and below its vertex, being given in position, to describe the parabola which will pass through that point. BOOK I.

Fig. 8.

The axis FH, its vertex F, and D a point without it, and below the vertex F, being given in position; it is proposed to describe the parabola which shall pass through D.

Draw from the point D, GD perpendicular to the axis, find (11. 6. Elem.) FK a third proportional to the two straight lines FG, GD; then, taking FB equal to the fourth part of it, and making FC equal to FB, draw BA parallel to DG; and let a parabola be described, having C for its focus, and AB for the directrix; this parabola will pass through the point D. For since FG, GD, FK are proportionals, the square of GD is equal to the rectangle GFK: and FK is the parameter of the (def. 6. 1.) diameter FG; therefore the point D (1. cor. 13. 1.) is in the parabola.

PROP. XVIII. Prob.

Fig. 9.

Two straight lines AB, AC, which meet each other in the point A, being given in position, and a straight line DE being given in magnitude; to describe a parabola which may have AB for a diameter, and DE for the parameter of AB, and which the straight line AC may touch at the point A.

Take DF the fourth part of DE, and in BA produced, make AG èqual to DF, and draw GH at right angles to AG; then, making the angle CAK equal to GAC, and the straight line AK equal to AG, describe a parabola, which may have K for the focus, and GH for the directrix; AB will be one of its diameters, and the parameter of AB will be equal to the (def. 4. 6.) quadruple of AG, that is, to DE: and since AG is equal to AK, the point A is in the parabola; and since the angle GAC is equal to CAK, the straight line AC touches the parabola at the point A.

Cor. If from a point L, a straight line LM be drawn making a given angle LMA, with a straight line AB given in position; and if the square of LM be equal to the rectangle contained by a given straight line DE, and the segment MA, intercepted between the same LM and the given point A; the point L is in a parabola given in position. Draw through the point A a straight line AC parallel to LM, describe, according to the proposition, a parabola which shall have AB for a diameter, and DE for the parameter of AB, and which the straight line AC may touch in the point A; this parabola is the *locus* of the point L: for since the square of LM is, by hypothesis, equal to the rectangle contained by MA and DE, and that LM is parallel to AC which touches the parabola, and consequently to straight lines ordinately applied to the diameter MA; the point L is in the (1. cor. 13. 1.) parabola.

PROP. XIX. PROB.

Fig. 9

A diameter AB, and its vertex A, being given in position, and a straight line LM, which meets AB in M below the vertex, being given in position and magnitude; to describe a parabola which may pass through the point L, and in which the straight line LM may be ordinately applied to the diameter AB.

Through the vertex A draw AC parallel to LM, and let DE be a third proportional to AM, ML; and, according to the preceding proposition, describe a parabola which having AB for a diameter, and DE for the parameter of AB, and which AC may touch in the point A: then, because ML is parallel to AC, it is ordinately applied to the diameter AB; and because AM, ML, DE are proportionals, the square of ML is equal to the rectangle contained by the abscissa AM, and DE the parameter of the diameter AB; and therefore the point L is (1. cor. 13. 1.) in the parabola.

PROP. XX. Prob.

A diameter of a parabola, and the vertex of that diameter, being given in position, and the *latus rectum* of the same diameter being given in magnitude, and a point in the parabola being given: to describe the parabola.

Fig. 10.

Let AB be the diameter given in position, and A its vertex; in AB, and above the vertex A, place the straight line AC equal to the given *latus rectum;* and let D be the given point in the parabola. Suppose what is required done; and let AD be the parabola to be described: and having drawn the straight line AE touching it in A, and meeting the diameter drawn through D in the point E, complete the parallelogram AEDF: therefore DF is ordinately applied to the diameter AB; and therefore the square of DF, or AE, is equal to the rectangle FAC: as, therefore, FA or DE to AE, so is AE to AC; and they contain the equal (29. 1. Elem.) angles DEA, EAC; therefore the triangle DEA is equiangular to the triangle EAC: and thus the angle AEC is equal to the angle EDA, or FAD. If, then, upon AC a segment of a circle be described, containing an angle equal to FAD, the point E will (converse of 21. 3. Elem.) be in the circumference of this segment. But the angle FAD is given, because FA, DA are (26. dat.) given in position; therefore the angle AEC is given: and AC is given in position and magnitude; therefore the segment AEC (8. def. dat.) is given in position. The point E, then, is in the circumference of a circle given in position: but it is also in the straight line DE which is given in position; the point E, therefore, is given: and the point A is given: therefore the straight line AE is given in position. It is possible, therefore, to describe (18. 1.) a parabola which may have AB for a diameter, and AC for the *latus rectum* of AB, and which AE may touch in the point A.

In order to the composition, it is required, that a segment of a circle containing an angle equal to FAD, be described upon AC, and that a straight line drawn through the point D, parallel to AB, meet the circumference of that segment. But these conditions it is sometimes impossible to fulfil. Hence the problem cannot always be solved.

When the straight line drawn through D parallel to AB, is a tangent to the segment, the problem admits of only one solution. *

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* In all cases in which the straight line drawn through D parallel to AB cuts the circumference of the segment in two points, the problem admits of two solutions.

PLATE II.

Fig. 5.

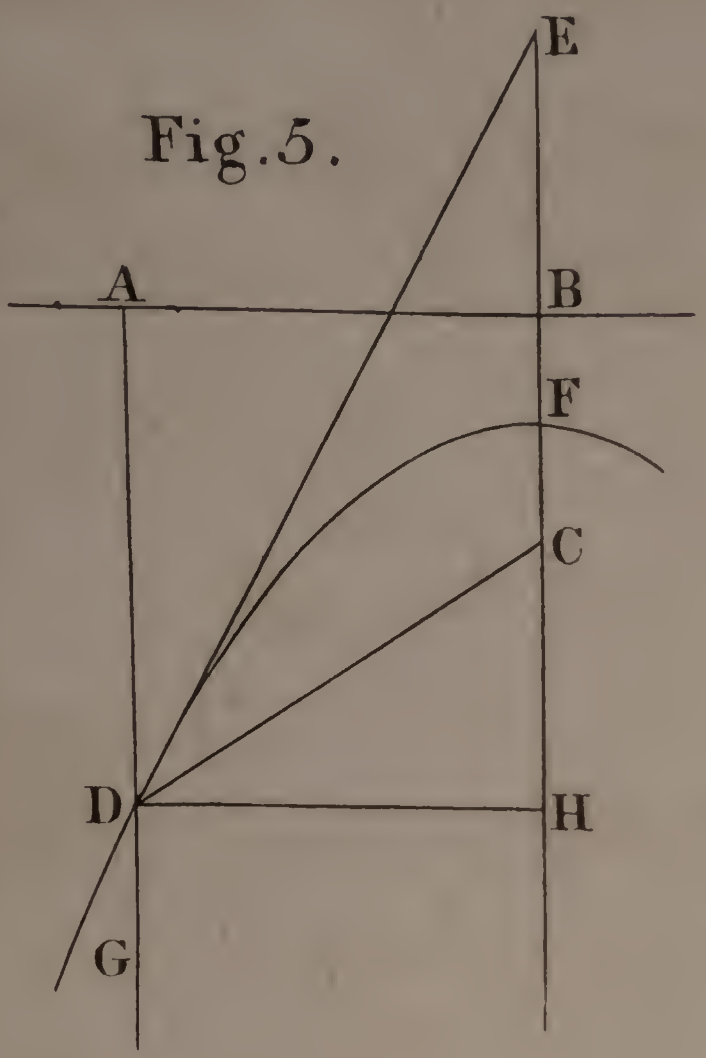


Fig. 6.

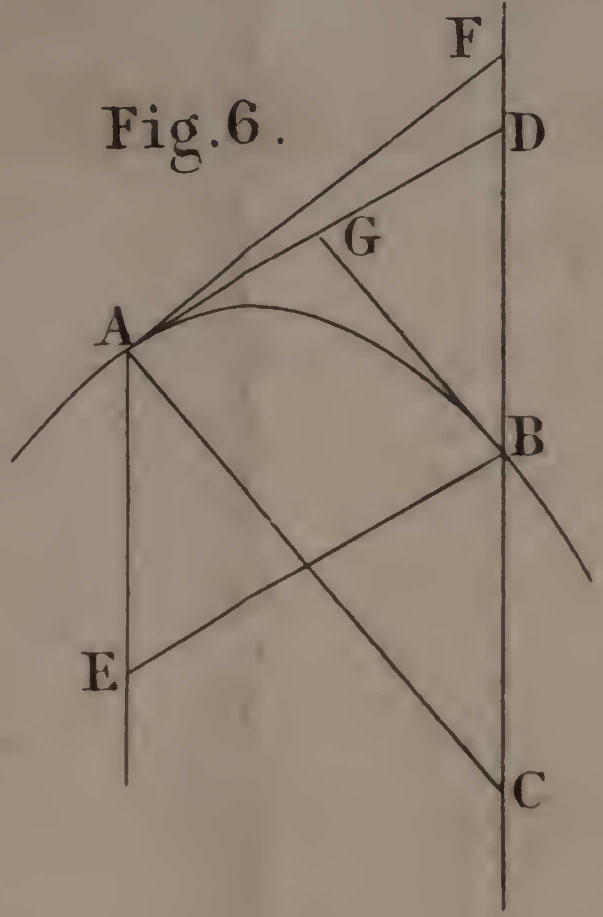


Fig. 7.

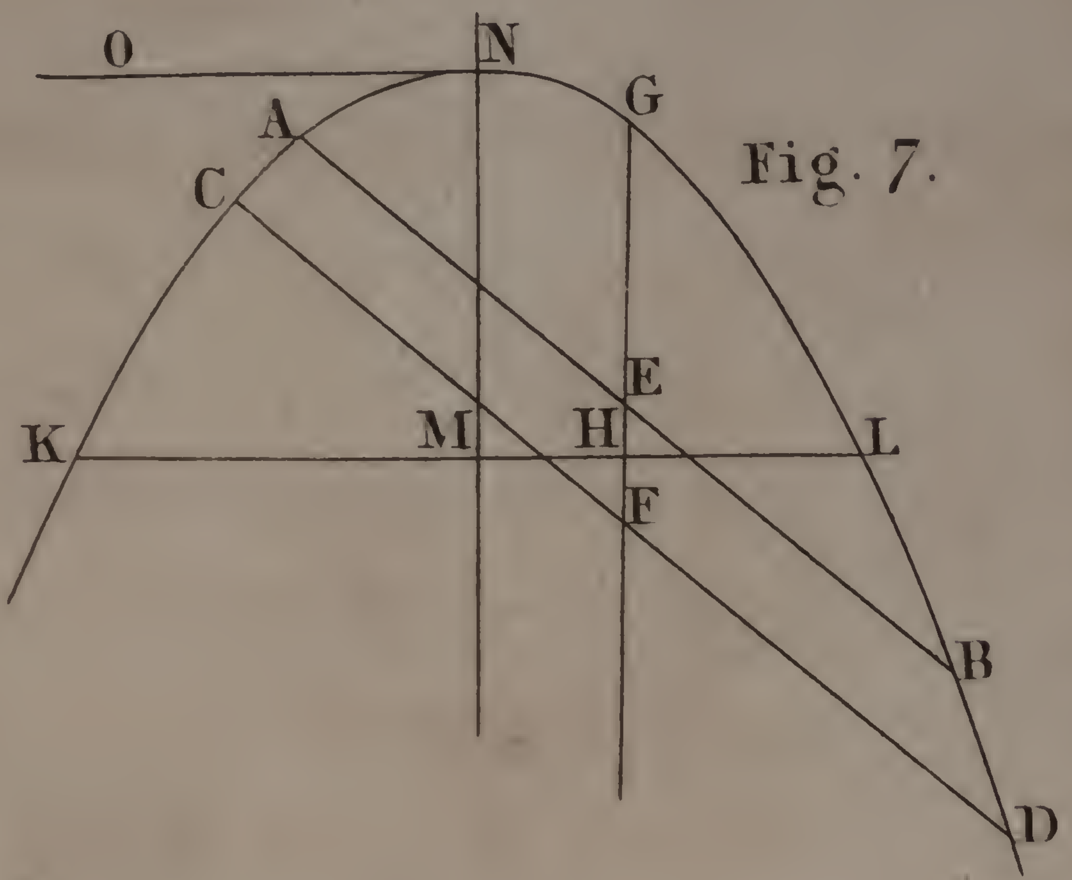


Fig. 8.

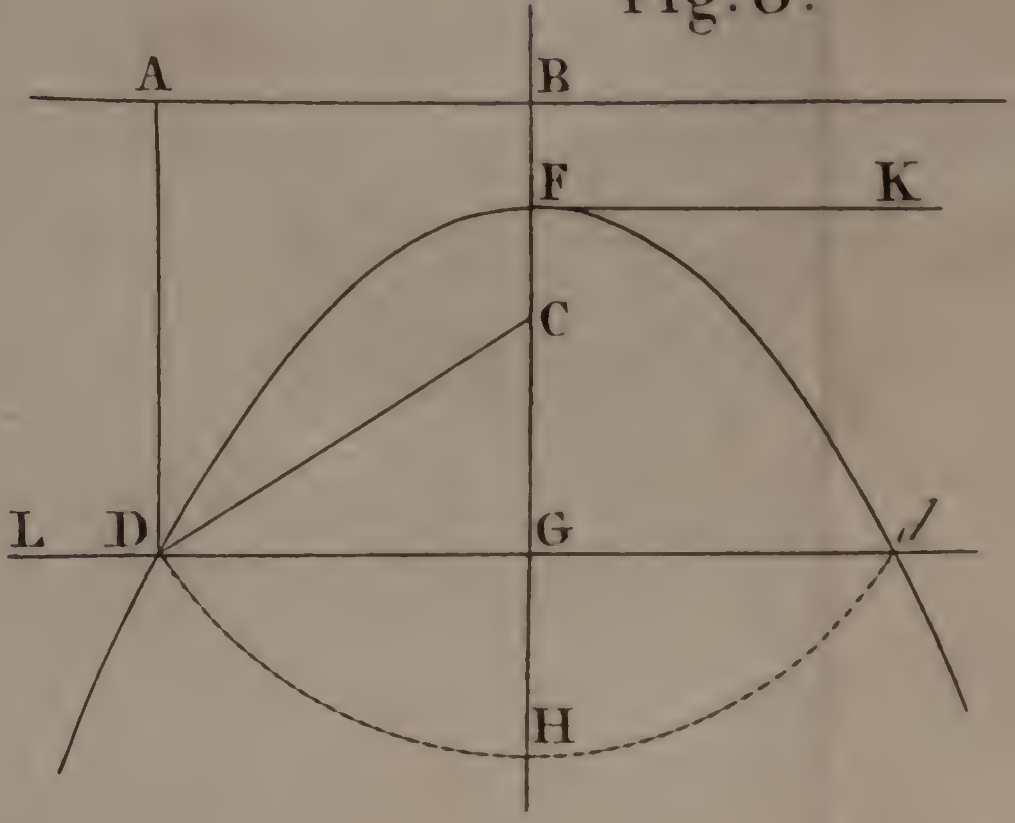


Fig. 10.

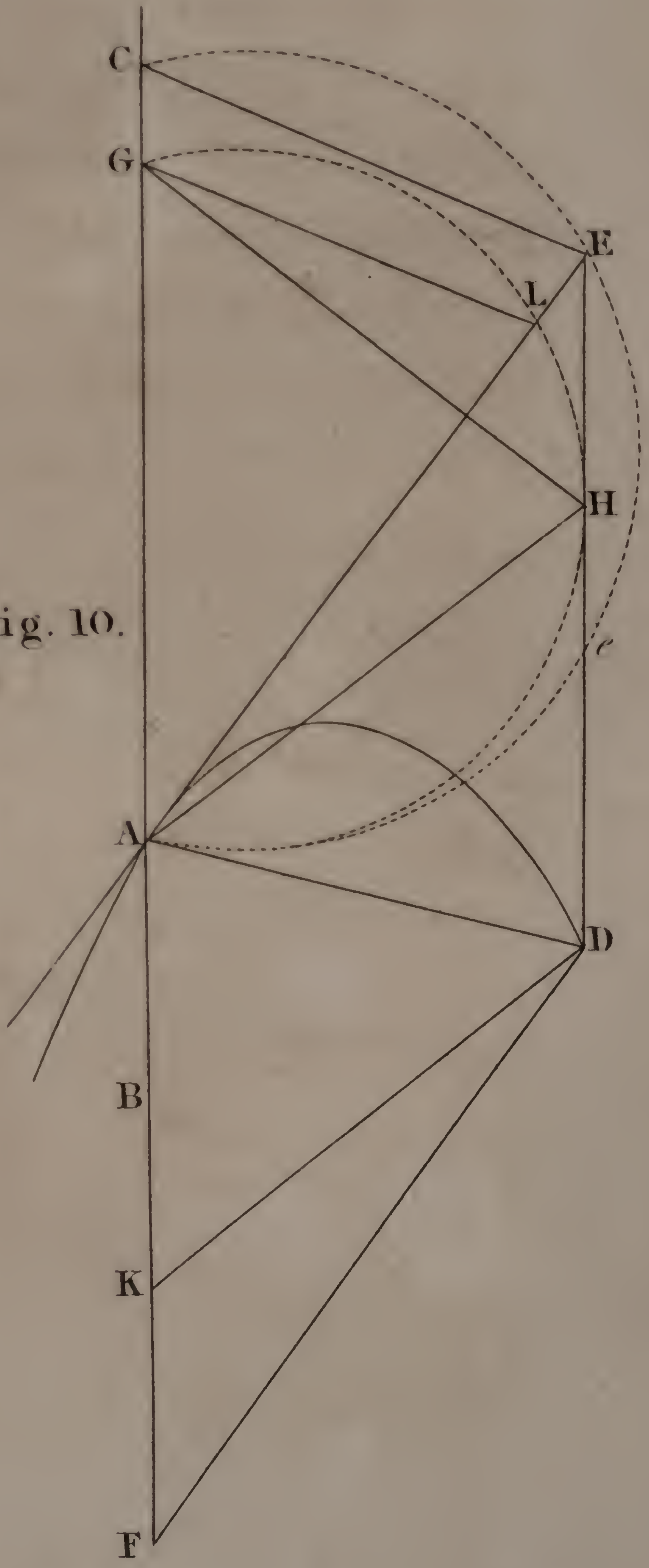
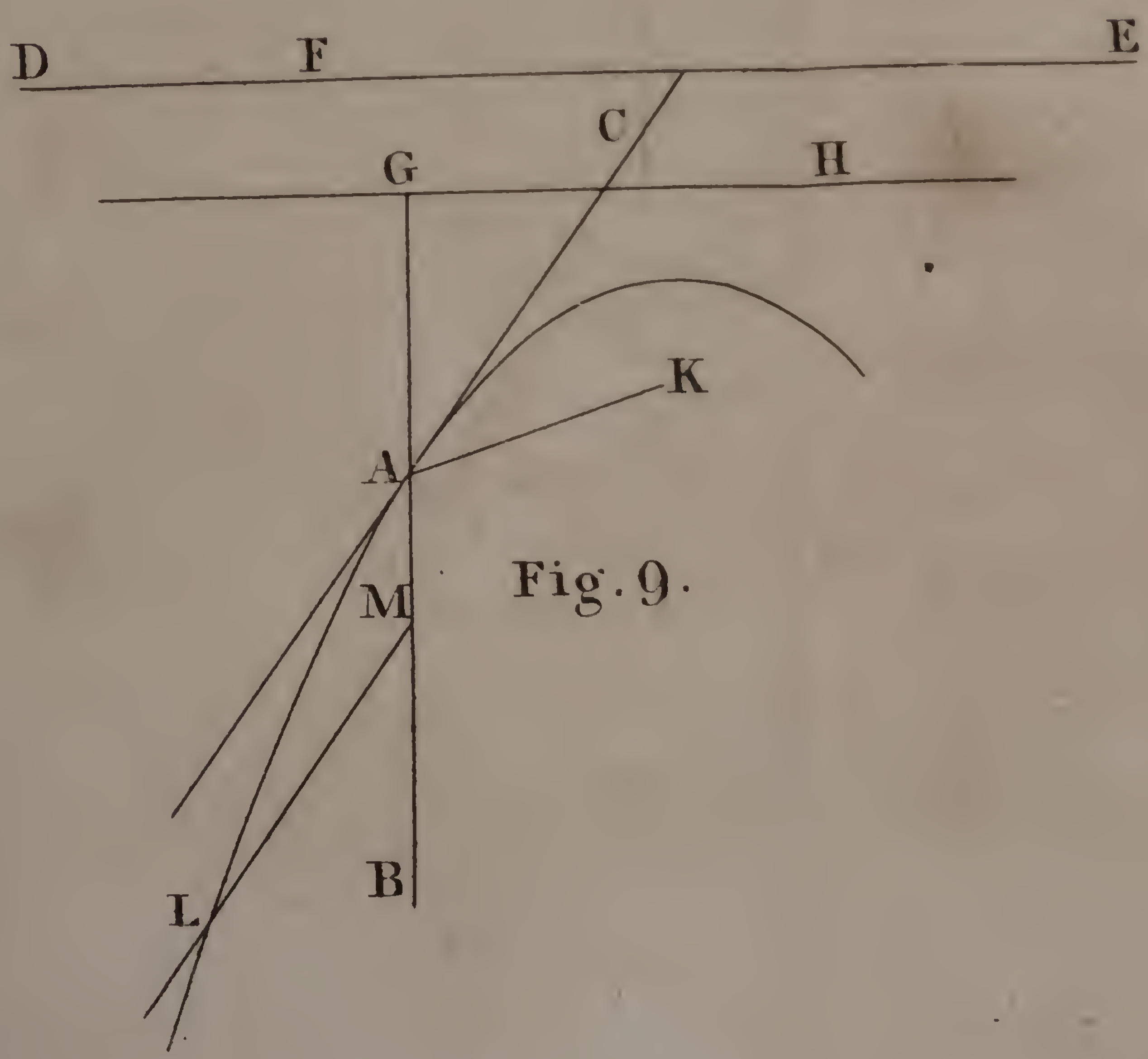


Fig. 9.



The parabola, and *latus rectum* of the diameter AB, which solve this case, are determined, if a parabola be found having AB for a diameter, and A for the vertex of AB, and which passes through the point D; and if the *latus rectum* of BA be such, that a segment of a circle described upon it, when placed above A, and in the direction of AB, may contain an angle equal to BAD; and that a straight line drawn through D, so as to be parallel to AB, may touch the circumference of that segment: suppose what is required done: let AG be the parameter of the diameter AB; and upon AG let the segment of a circle containing an angle equal to the angle BAD, or ADE, be described; and let the straight line DE parallel to AB touch the circumference of that segment in H, and join AL, GH: since, then, the angle AHD is equal (32. 3. Elem.) to AGH in the opposite segment, and that, according to the hypothesis, the angle ADH is equal to AHG; the triangles ADH, AHG are equiangular: the angle DAH is, therefore, equal to HAG: but the angle DAG is given; and, consequently, its half DAH is given: and the straight line AD is given in position: therefore AH also is (29. dat.) given in position: the point H too is given, where AH meets DE given in position; and the angle AHG is given: hence HG is given (29. dat.) in position; and therefore the point G is given: hence the straight line AG is given in magnitude: since, then, AG in the parabola, which passes through the point D, is the *latus rectum* of the diameter AB, of which A is the vertex; because a straight line touching the parabola in the vertex of the diameter AB, meets, as hath been proved, the diameter drawn through D, in the point where this diameter meets the circumference of the circle, the segment of which, described upon the *latus rectum* of the diameter, passing through A, contains an angle equal to ADH; and since, in the present case, the diameter DE meets the circumference of this segment in H; therefore HA touches the parabola in A: and AB, AH being given in position, and AG given in magnitude, the parabola, according to the 18th proposition, can be described.

The composition of this case is as follows:

Join AD, and through D draw DE parallel to AB; to DE draw AH, bisecting the angle DAG: and through H to AB draw HG, making the angle AHG equal to ADH or DAB; and let a parabola be described which may have AB for a diameter, and AG for the *latus rectum* of AB; and which AH may (18. 1.) touch in A: this parabola will pass through D, and DH will touch the circle described about AHG. Draw DK parallel to AH; and since the triangles DAH, AHG are isosceles and equiangular, DH, HA, AG, and consequently KA, KD, AG are proportionals; the square of DK is, therefore, equal to the rectangle KAG; and DK is parallel to the tangent AH: hence the point D is in the (1. cor. 13. 1.) parabola: and because the angle AHD is equal to AGH in the opposite segment, DH touches (conv. 32. 3. Elem.) the circle in H.

It remains to be inquired, whether the parameter AG be greater or less than the parameter of the diameter AB in any other parabola, having AB for a diameter, and A for the vertex of AB, and which passes through D. Let there be any other parabola admitting of these conditions; and let AE touch it in A, and meet the diameter passing through D in the point E, and the circle GHA in L: having joined LG, draw EC parallel to it; draw also DF parallel to EA; DF, therefore, is ordinately applied to the diameter AB: and because the angle ADE is equal to the angle AHG or ALG, that is, to AEC, and that the angle DEA is equal to EAC, the triangle EDA is equiangular to the triangle AEC: therefore the straight lines DE, EA, AC, that is, AF, FD, AC are proportionals; the square of DF, is, therefore, equal to the rectangle FAC: and for this reason, AC is the parameter of the diameter AB in this parabola. And because DE touches the circle in H, AL is less than AE; and therefore AG is less than AC: therefore AG is the least of all the possible parameters of the diameter AB, in parabolas which have AB for a diameter, and A for the vertex of AB, and which pass through D.

After the same manner it may be shown, that in any parabola whatever, which answers the conditions of the proposition, the *latus rectum* of the diameter AB is greater or less, according as the tangent drawn through A, and situated on either side of the tangent AH, is more remote from, or nearer to, the same AH.

To proceed to the composition of what was analysed in the first case: if the proposed parameter be equal to AG, found in the manner above mentioned, the parabola, as hath been shown, may be described, and will be the only one that can fulfil what is required in the problem. If, next, the proposed parameter be less than AG, it is impossible to construct the problem: or if the proposed parameter, for example, AC, be greater than AG; upon AC describe the segment of a circle containing an angle equal to ADH, or DAB; and since DH touches the circle AHG, it must cut the segment described upon AC in two points: let E be one of them; and join AE; and let a parabola (18. 1.) be described, having AB for a diameter, and AC for the parameter of AB, and to which the straight line AE may be a tangent; and draw DF parallel to AE; then it may be shown, as above, that DE, EA, AC, that is, that AF, FD, AC, are proportionals; and therefore the square of DF is equal to the rectangle FAC, contained by the abscissa FA and the parameter AC; and that, consequently, the parabola passes through the point D. The same thing may be demonstrated with regard to the other parabola, which has for a tangent the straight line joining A, and the other point of intersection *e*. And as, in the investigation of the problem, it has been proved, that the angles GAH, HAD are equal; the angle AKD is, therefore, equal to ADK; and, of consequence, AK is equal to AD: but AG is a third proportional to AK, KD, or to AD, DK; that is, the least parameter is a third proportional to the straight line which joins the vertex of the diameter given in position and the given point; and the straight line which is drawn from the same point to the diameter, so as to cut off from the diameter a segment equal to the first proportional.

The first nine definitions in the first book of Apollonius of Perga's Conic Sections.

Ap. Def.

1. VIII. If a straight line joining any point and the circumference of a circle not in the same plane with the point, be produced from the point in the opposite direction, and then, while the point remains fixed, be carried round in the direction of that circumference till it return to the place from whence the motion commenced; by the revolution of that straight line, a surface, called *the conical surface*, and which consists of two surfaces connected together at the fixed point, will be described. The two connected surfaces may each of them be infinitely increased, if the straight line with which they are described be produced both ways to an infinite distance.

2. IX. The fixed point is called *the vertex* of the conical surface.

3. X. The straight line drawn through the point and the centre of the circle is called *the axis.*

4. XI. The figure contained by the circle, and the surface which is intercepted between the vertex and the circumference of the circle, is called *the cone.*

5. XII. The same fixed point, which is the vertex of the surface of the cone, is named *the vertex of the cone.*

6. XIII. The straight line drawn from the vertex to the centre of the circle, is called *the axis of the cone.*

7. XIV. And the circle itself is named *the base of the cone.*

8. XV. Cones which have their axis at right angles to the base, are called *right-angled cones.*

9. XVI. And cones which have not their axis at right angles to the base, are called *scalene cones.*

PROP. XXI. *(Prop. 1. B. 1. Apoll.)*

Straight lines drawn from the vertex of the surface of a cone to points in that surface, are in that same surface.

Fig. 11. Let there be the surface of a cone: let A be its vertex; and having taken any point B

in that surface, join AB: the straight line AB is in that same surface.

For, if possible, let ACB be a straight line drawn from the vertex A to the point B, and which is not in the surface of the cone; and let DE be the straight line with which the cone is described, and the circle EF the base; and if DE be revolved in the circumference of EF, it will pass through the point B and the vertex A; and thus two straight lines ACB, AGB will have the same extremities: which is absurd. Therefore the straight line drawn from the point A to B, is not without the conical surface; therefore it is in that surface.

Cor. A straight line drawn from the vertex of a cone to any point within the surface, falls within the surface; but if drawn from the vertex to any point without the surface, it falls without the surface.

PROP. XXII. *(Prop. 3. B. 1. Apoll.)*

If a cone be cut by a plane passing through its vertex, the section is a triangle.

Fig. 12.

Let there be a cone which has the point A for its vertex, and the circle BDC for its base; let it be cut through the point A by any plane; and let the sections made in the surface be the lines AB, AC, and the section in the base the straight (3. 11. Elem.) line BC; ABC is a triangle.

For since the straight line drawn from the point A to B, is both in the cutting plane and in (21. 1.) the conical surface, it is the common section of the two; therefore the section AB is a straight line: for a like reason, the section AC is a straight line; and BC too is a straight line: therefore the section ABC is a triangle.

PROP. XXIII. *(Prop. 4. B. 1. Apoll.)*

If the conical surface on either side of the vertex be cut by a plane parallel to the circle which is the base of the cone; the common section of this plane with the conical surface is a circle having its centre in the axis; and the figure contained by this circle, and that part of the conical surface which is intercepted between it and the vertex, is a cone.

Fig. 13.

Let there be a conical surface the vertex of which is A, BC being the circle in the circumference of which the straight line revolves which describes the *surface;* let it be cut by any plane parallel to the circle BC, and let this plane make in it a section DLE: The line DLE is the circumference of a circle the centre of which is in the axis. Take the centre of the circle BC, and let it be F; join AF; AF, consequently, is (def. 10.) the axis, and meets the cutting plane; let it meet it in G; next, let any plane pass through the same AF; and the section made by this plane will be (22. 1.) a triangle ABC. And because the points D, G, E are all in the cutting plane DLE, and in the plane ABC, DGE is (3. 11. Elem.) a straight line. Again, in the line DLE take any point H; join AH, and produce it; AH then will (21. 1.) meet the circumference BC; let it meet it in K, and join GH, FK: and because the two parallel planes DLE, BC are cut by the plane ABC, their (16. 11. Elem.) common sections with it are parallels: DE, consequently, is parallel to BC; and, for the same reason, GH is parallel to FK: therefore, as AF to (4. 6. Elem.) AG, so is FB to GD, FC to GE, and FK to GH; and the three straight lines BF, KF, CF are equal; therefore the three straight lines DG, GH, GE (14. 5. Elem.) are also equal. After the same manner it may be demonstrated, that any other straight lines whatever, drawn from the point G to the line DLE, are equal. The line DLE is, therefore, the circumference of a circle having its centre G in the axis.

Cor. The figure contained by the circle DLE, and that part of the conical surface which is intercepted between this circle and the point A, is a cone; and the common section of the cutting plane and the triangle passing through the axis, is a diameter of the circle DLE.

PROP. XXIV. *(Prop. 5. B. 1. Apoll.)*

If a scalene cone cut through the axis by a plane at right angles to the base, be cut also by another plane at right angles

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to the triangle passing through the axis; if this other plane cuts off, towards the vertex, a triangle similar to the triangle through the axis, both triangles being in one plane, but sub-contrarily situated; the section made in the cone by this other plane is a circle.

Fig. 14.

Let there be a scalene cone, the vertex of which is the point A, and the base the circle BLC; let it be cut through the axis by a plane perpendicular to the base, and let the section be the triangle ABC; let it be also cut by another plane at right angles to the triangle ABC; and let this other plane cut off, towards the vertex, the triangle AGK similar to the triangle ABC, but * sub-contrarily situated; and let the section made in the surface be the line GKH: This line is the circumference of a circle.

In the lines GHK, BLC take certain points

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* The meaning is this: the base GK is to be so placed, that it may make the angle AKG, and not the angle AGK, equal to the angle ABC.

H, L, from which let perpendiculars be drawn to the plane of the triangle ABC; these perpendiculars will (38.11. Elem.) fall on the common sections of the planes: accordingly, let them be HF, LM: HF, therefore, is parallel to (6. 11. Elem.) LM: next, through F draw DFE parallel to BC: the plane, therefore, which passes through FH, DE is parallel to the (15. 11. Elem.) base of the cone; and, for this reason, the section DHE is (23. 1.) a circle, of which DE is a diameter: the rectangle, therefore, contained by DF, FE is (35. 3. Elem.) equal to the square of FH. And since ED is parallel to BC, the angle ADE is equal to the angle ABC; and the angle AKG is placed equal to the angle ABC; therefore the angle AKG is also equal to ADE: and the angles at F are equal, for they are opposite vertical angles; therefore the triangle DFG is similar to the triangle KFE: therefore as EF to FK, so is GF to FD; therefore the rectangle EFD is equal to the rectangle KFG. But the rectangle EFD, (that is, the rectangle contained by DF, FE,) has been proved to be equal to the square of FH; therefore the rectangle contained by KF, FG is equal to the same square of FH. It may, in like manner, be demonstrated, that the square of any straight line whatever, drawn from the line GHK, so as to be perpendicular to GK, is equal to the rectangle contained by the segments into which that straight line divides the same GK: the section GHK is, therefore, a circle having GK for a diameter.*—A section of this kind may be named *a sub-contrary section.*

PROP. XXV.

If a cone cut through the axis by a plane, be cut likewise by another plane, cutting its base in the direction of a straight line perpendicular to the base of the triangle passing through the axis; and if the common section of the triangle through the axis, and of the plane cutting the base of the cone in the direction of the perpendicular, be parallel to one

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* See the *lemma* placed at the end of this book.

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of the sides of the triangle through the axis; the line which is the common section of the plane cutting the base, and of the conical surface, is a parabola, having for a diameter the straight line which is the common section of the triangle through the axis, and of the same plane cutting the base.

Fig. 15.

Let there be a cone, the vertex of which is the point A, and the base the circle BC; let it be cut through the axis by a plane, and let the section be the triangle ABC; let it be also cut by another plane, cutting its base in the direction of the straight line DE perpendicular to the straight line BC; let the line DFE be the section made in its surface; and let FG, the common section of the triangle through the axis, and that other plane, be parallel to AC, one of the sides of that triangle: the line DFE is a parabola, and FG one of its diameters.

In the section DFE take any point H, and through H draw HK parallel to DE to meet

FG in K; and through K draw LM parallel to BC: therefore the plane passing through HK, LM is (15. 11. Elem.) parallel to the plane through DE, BC, that is, to the base of the cone: and, consequently, the plane through HK, LM is a (23. 1.) circle of which LM is a diameter. But HK is perpendicular to LM, (10. 11. Elem.) because DE is perpendicular to BC: therefore the rectangle LKM is equal to the square of HK (35. 3. Elem.); and, in like manner, the rectangle BGC is equal to the square of DG: therefore the square of DG is to the square of HK, as the rectangle BGC to the rectangle LKM; and GC is equal to KM; consequently the rectangle BGC is to the rectangle LKM, as BG to LK, that is, as GF to KF: therefore the square of DG is to the square of HK as the straight line GF to the straight line KF. Let a parabola (19. 1.) be described, which may have GF for a diameter, and F for the vertex of GF, and in which DG may be ordinately applied to the same GF: and because the point D, by construction, is in the parabola described, the point H is likewise in this same parabola (2. cor. 13. 1.). And the same thing may be demonstrated with regard to all the points of the section DFE.

The second Lemma of Pappus, as it is extant in the first book of Apollonius's Conic Sections.

Fig. 16. Let ABC be a line, and let AC be a straight line given in position; and let all the straight lines drawn from the line ABC, so as to be at right angles to AC, be such, that each of them may have its square equal to the rectangle contained by the segments, into which it cuts AC; ABC is the circumference of a circle, and AC a diameter of that circle.

From the points D, B, E draw perpendiculars DF, BG, EH: then the square of DF is equal to the rectangle AFC, the square of BG to the rectangle AGC, and the square of EH to the rectangle AHC. Bisect AC in K, joining KD, KB, KE: then, since the

PLATE III.

Fig. 11.

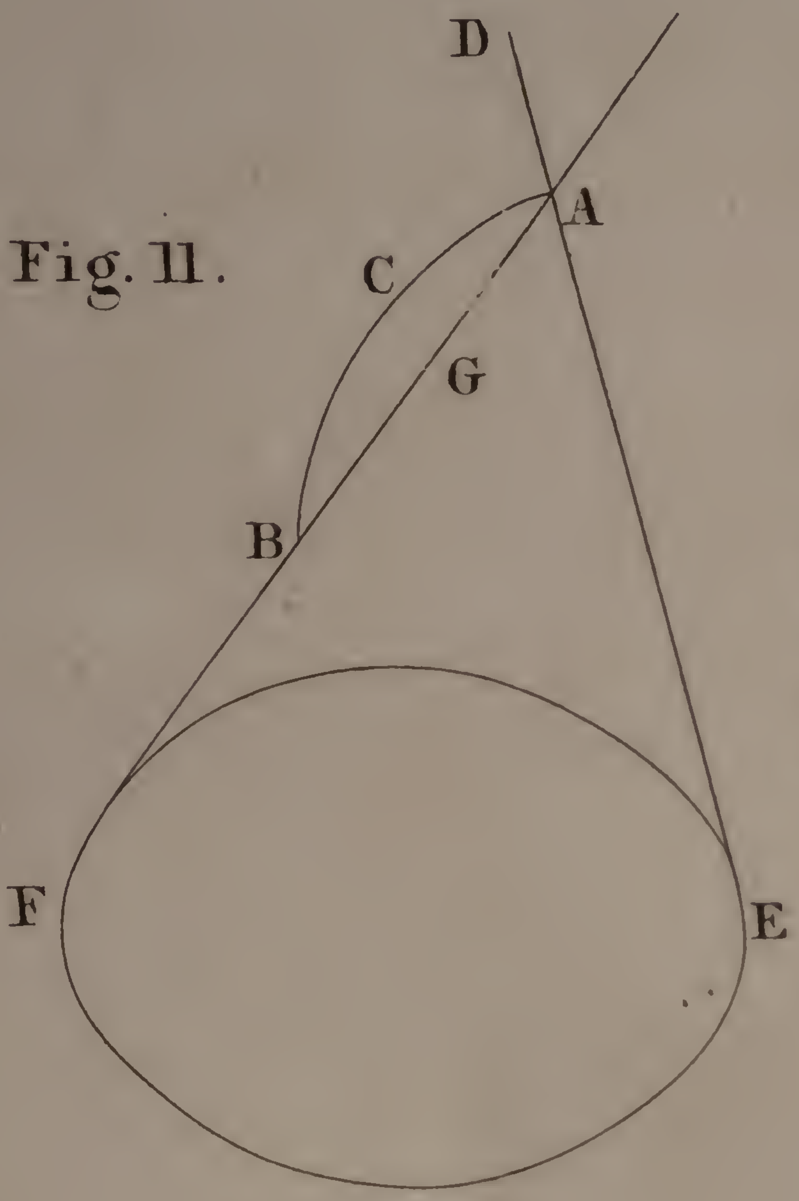


Fig. 12.

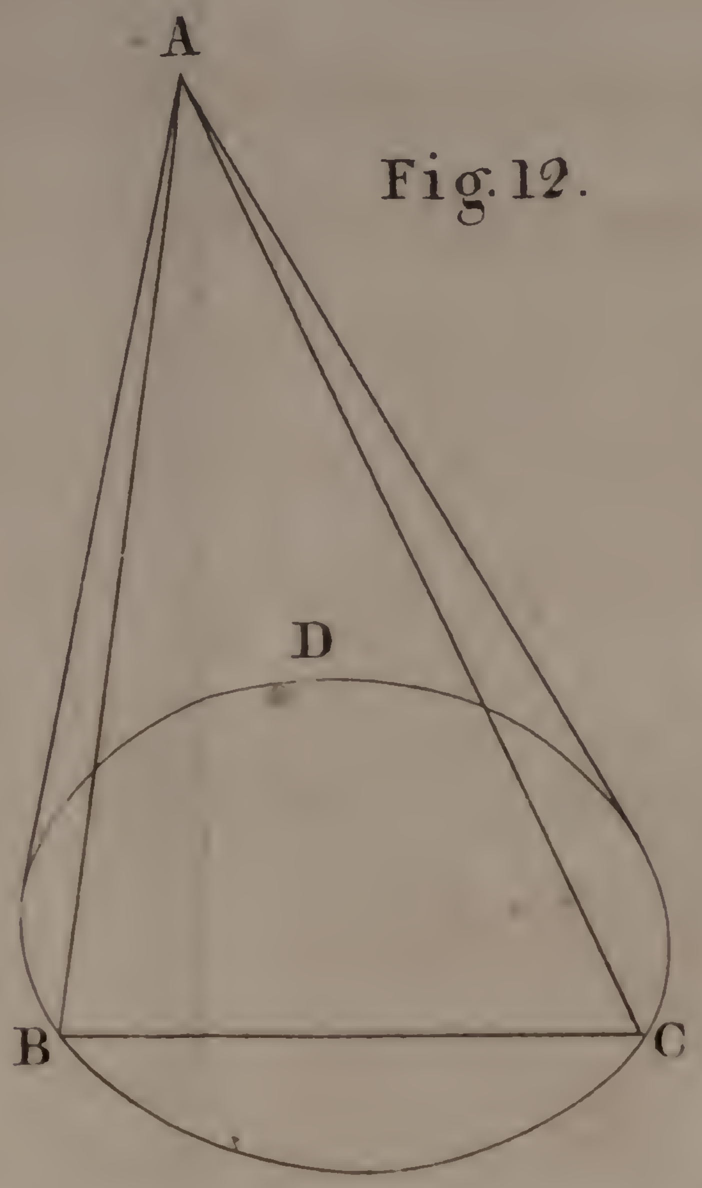


Fig. 13.

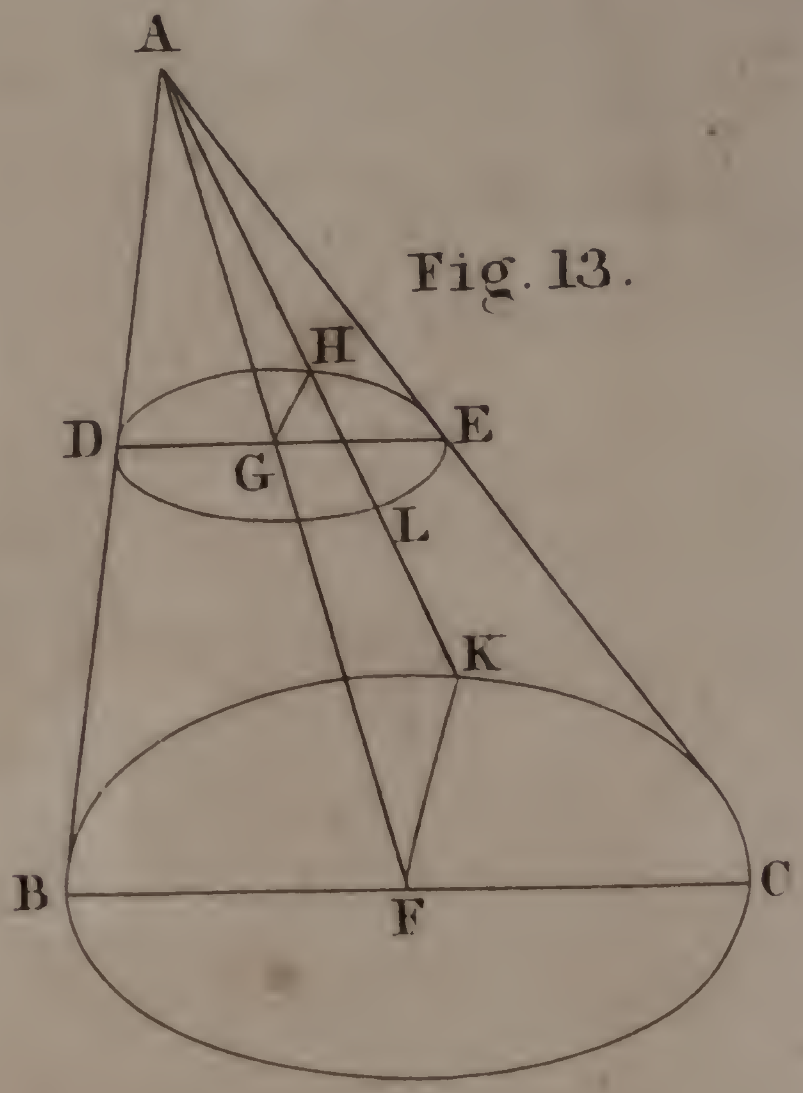


Fig. 16.

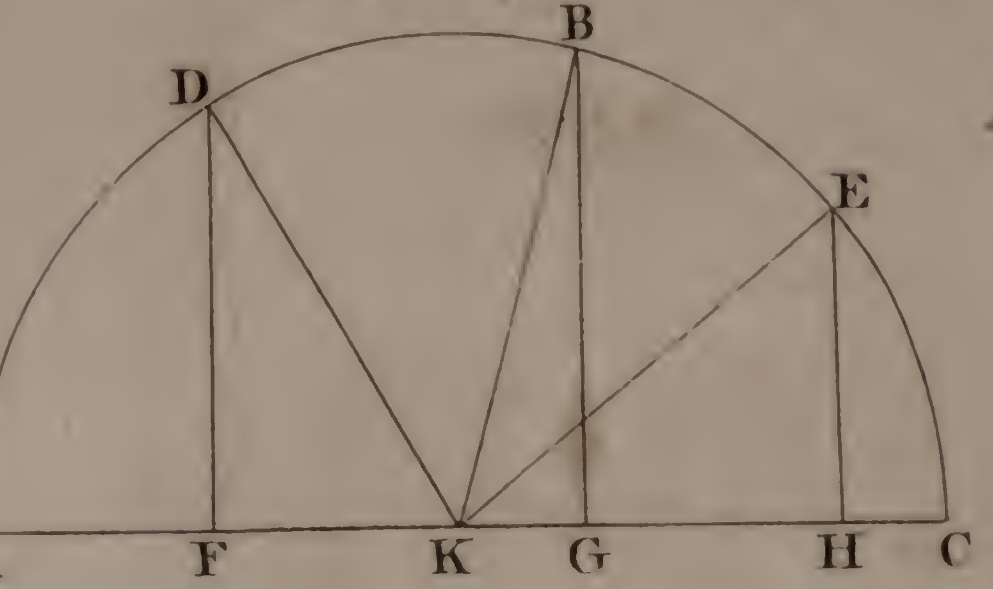


Fig. 14.

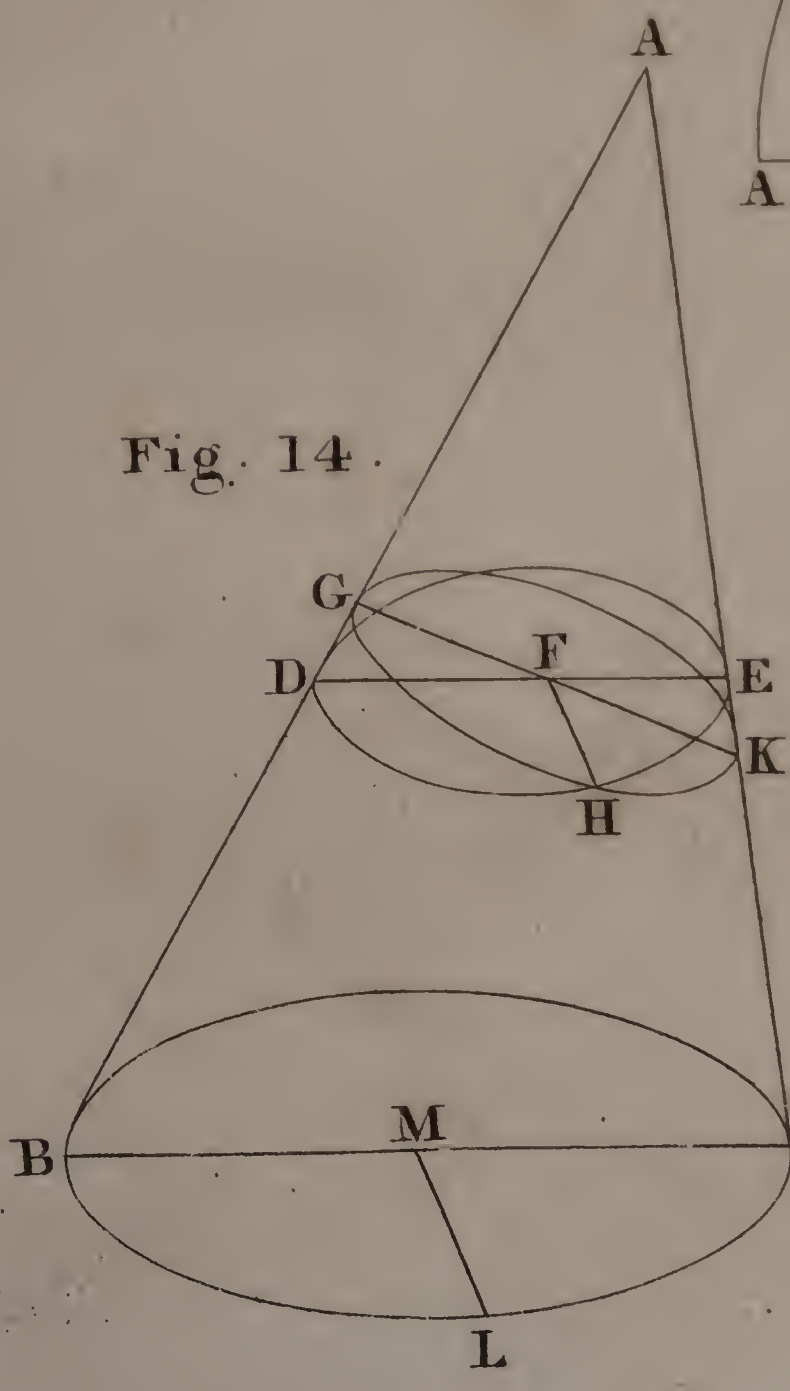
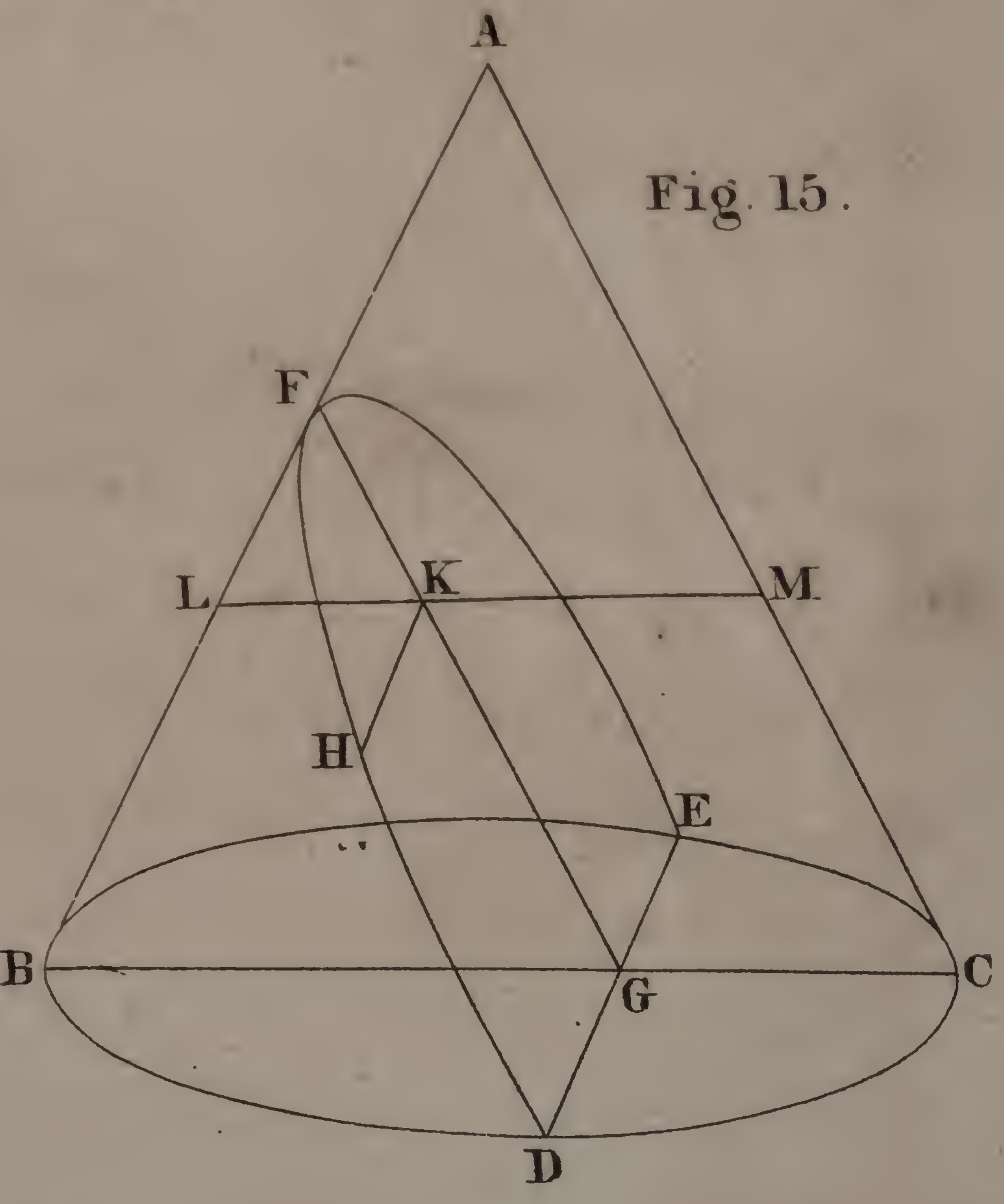


Fig. 15.



rectangle AFC, together with the square of FK, is equal (5. 2. Elem.) to the square of AK, and that the square of DF is, by hypothesis, equal to the rectangle AFC; the square of DF, together with the square of FK, is equal to the square of AK: consequently, the square of DK (47. 1. Elem.) is equal to the same square of AK: AK, also, is equal to KD. In like manner, each of the straight lines BK, EK may be proved to be equal to AK or KC; consequently ABC is the circumference of a circle which has the point K for its centre, and is described about AC as a diameter.

ELEMENTS

OF THE

CONIC SECTIONS.

Book Second.

OF THE ELLIPSIS.

DEFINITIONS.

BOOK II. Fig. 1. n. 1. I. IF in two points D, E, taken in a plane, are fixed the ends of a string, the length of which is greater than the distance between these points; and if the point of a pin H applied to the string, and held so as to keep it uniformly tense, is moved round, till it return to the place from whence the motion began :

the point of the pin, as it moves round, describes upon the plane a line called the ELLIPSIS.

II. The points D, E are named the *foci.*

III. The point C which bisects the straight line between the foci, is named the *centre* of the ellipsis.

IV. A straight line passing through the centre, and terminated both ways by the ellipsis, is named a *diameter;* and the points where a diameter meets the ellipsis, are named the *vertices* of that diameter.

V. The diameter which passes through the foci, is named the *greater axis.*

VI. The diameter perpendicular to the greater axis, is named the *lesser axis.*

VII. Two diameters, each of which bisects all straight lines in the ellipsis that are parallel to the other, are named *conjugate diameters.*

VIII. A straight line not passing through the centre, but terminated both ways by the ellipsis, and bisected by a diameter, is said to be *ordinately applied* to that diameter, or it is named, simply, an *ordinate* to that diameter. Also a diameter parallel to a straight line ordinately applied to another diameter, is said to be *ordinately applied* to that other diameter.

IX. A third proportional to two conjugate diameters is called the *latus rectum*, or the *parameter*, of that diameter, which is the first of the three proportionals.

X. A straight line which meets the ellipsis only in one point, is said to touch it in that point.

PROP. I. THEOR.

If two straight lines be drawn from any point in an ellipsis to the foci, they are together equal to the greater axis.

Fig. 1.

The two straight lines HD, HE drawn from H, a point in an ellipsis, to the foci D, E are together equal to AB the greater axis.

Because H is a point in the ellipsis, HD, HE are together equal to the length of the string with which it is described; and because the point A is likewise in the ellipsis, DA, EA are together equal to the length of the string. For a like reason, EB, DB are together equal to the same length: DA, EA are, therefore, equal to EB, DB. Take away the common part DE, and the remainder, twice AD, will be equal to the remainder twice EB: AD, therefore, is equal to EB. Add the common part AE; and AD, together with AE, will be equal to the greater axis: but AD, together with AE, is equal to the length of the string, that is, to HD together with HE; therefore HD and HE are together equal to the greater axis AB.

Cor. 1. The greater axis is bisected in the centre C. For since (def. 3.) DC is equal to EC, and that DA is equal to EB; AC is equal to CB.

Cor. 2. Two straight lines drawn from a point without an ellipsis to the foci, are together greater than the greater axis; but if drawn from a point within an ellipsis to the foci, they are (20. 1. Elem.) together less than that axis.

Cor. 3. A point is either in, without, or within, an ellipsis, according as two straight lines drawn from it to the foci are either equal to, greater, or less, than the greater axis.

Cor. 4. The distance of either vertex F, or

G of the lesser axis from either of the foci, is equal to half the greater axis. Join GD, GE: then, because CD is equal to CE, and CG common, and the angles at C right angles; the triangle CDG is equal to the triangle CEG; therefore DG is equal to EG: but DG and EG are together equal to the greater axis; therefore each of them is equal to the half of it.

Cor. 5. The lesser axis FG is bisected in the centre. Draw straight lines DF, DG from the focus D to the vertices of the lesser axis; then DF, DG, by the preceding corollary, will be equal; the angle DFC is, therefore, equal to DGC, and (def. 6.) DCF, DCG are right angles; therefore CF is (26. 1. Elem.) equal to CG.

PROP. II. Theor.

The square of half the lesser axis is equal to the rectangle contained by the segments of the greater axis, intercepted between the vertices of that axis and either of the foci.

From either focus as E to either vertex of the lesser axis as G, draw the straight line GE; let C be the centre of the ellipsis, and A, B the vertices of the greater axis: then the squares of GC, CE are together equal to the square of GE, that is, to the square (4. cor. 1. 2.) of CB, that is, to the rectangle AEB together with the square of EC (5. 2. Elem.): take away the common square of CE, and the remaining square of GC will be equal to the remaining rectangle AEB. Fig. 1.

PROP. III. Theor.

Every diameter of an ellipsis is bisected in the centre.

Fig. 2. Let HK be a diameter; it is bisected in the centre C: for if CK be not equal to CH, let C*k* be equal to CH; and from the points H, K, *k* draw straight lines to the foci D, E: then, because CD is equal to CE, and that C*k* is made equal to CH; the triangle DCH is (4. 1. Elem.) equal to the triangle EC*k*, and the base DH to the base E*k*. In the same manner, EH is shown equal to D*k*: therefore

Ek, Dk are together equal to DH and HE together, that is, to EK and DK together: which is (21. 1. Elem.) absurd; therefore CH is equal to CK.

PROP. IV. THEOR.

If a straight line be drawn from a point in an ellipsis, at right angles to the greater axis; and if another straight line be drawn from the same point to the nearest focus; half the greater axis is to the distance of this focus from the centre, as the distance between the centre and the perpendicular is to the excess of half the greater axis above the straight line drawn to this same focus.

Fig. 1. II. 1. 2. From H, a point in an ellipsis, let HK be drawn perpendicular to the greater axis AB, HE being drawn from the same point to the focus E: then CB, the half of the greater axis, is to CE, the distance of the focus E

from the centre, as CK, the distance between the centre C, and the perpendicular HK, is to the excess of CB above HE.

Having made BL equal to EH, and drawn HD to the other focus, describe from the centre H, distance HE, a circle meeting the axis AB again in O, and the straight line DH in the points M, N: then, because DE is the double of CE, and OE the double of (3. 3. Elem.) KE, the whole, or the remainder, DO, is double the whole, or the remainder CK: and because DN is equal to DH together with HE, that is, to (1. 2.) AB, therefore DN is the double of CB: but MN is the double of HE or LB; therefore the remainder DM is double the remainder CL: and because of the circle, the rectangle NDM is (cor. 36. 3. Elem.) equal to the rectangle EDO: therefore, as (16. 6. Elem.) ND or AB to DE, so is DO to DM: but the halves of magnitudes have the same ratio to one another which the wholes have: as therefore CB to CE, so is CK to CL, the excess of CB above LB, or HE.

Fig. 1. n. 1.

Cor. If in a straight line AB given in position and magnitude, and bisected in C, a point E be taken between the points B, C; and from a point H, KH be drawn perpendicular to AB; and HE be joined, and BL placed equal to it towards the point C: and if CL, CK, CE, BC be proportionals, and L, K be on the same side of C; the point H is in an ellipsis given in position, that is, in an ellipsis that has AB for the greater axis, and the point E for one of the foci.

Make AD equal to BE, and, as directed in the first definition, describe an ellipsis that shall have AB for the greater axis, and the points D, E for the foci; the point H will be in that ellipsis: for, if not, let KH meet the ellipsis in the point Q; join EQ, and make BR equal to it: therefore, by the proposition, as CR to CK, so is CE to CB; but by hypothesis, CL is to CK as CE to CB; therefore CR is to CK as CL to CK: CL, therefore, is equal to CR; which is absurd: the ellipsis, therefore, meets not the straight line KH in Q; nor, as may in like manner be proved, does it meet HK in any other point on the same side of AB than the point H. Therefore the point H is in the ellipsis.

PROP. V. THEOR.

Fig. 1. n. 1. 2.

The same construction remaining, if from the vertex of the greater axis nearest to H, the part BL be taken equal to the distance of H from the focus E; the square of the perpendicular HK, is equal to the excess of the rectangle AKB, contained by the segments into which the axis is divided in the point K, above the rectangle DLE, contained by the segments into which the distance of the foci is divided in the point L.

For since the straight line CB is cut into any two parts in the point L, the squares of BC, CL are together equal (7. 2. Elem.) to twice the rectangle BCL, together with the square of BL, that is, to twice (preced. prop. and 16. 6. Elem.) the rectangle ECK, together with the square of BL, or HE, that is, to twice the rectangle ECK, together with the squares of KE, KH, that is, to the (7. 2. Elem.) squares of EC, CK, and KH together: the squares, therefore, of BC, CL are together equal to the squares of EC, CK, and KH together: but the squares of BC, CL are together equal (5. 2. and 2. ax. Elem.) to the rectangle AKB, together with the squares of CK, CL, and the squares of EC, CK, and KH together are equal (5. 2. and 2. ax. Elem.) to the rectangle DLE, together with the squares of CL, CK, KH: therefore the rectangle AKB, together with the squares of CK, CL, are equal to the rectangle DLE, together with the squares of CL, CK, KH: from these equals take away the common squares of CK, CL, and there will remain the rectangle AKB, equal to the rectangle DLE, together with the square of HK; consequently, the square of KH is the excess of the rectangle AKB above the rectangle DLE.

PROP. VI. Theor.

If a straight line be drawn from a point of an ellipsis, perpendicular to either axis; the square of that axis is to the square of the other axis, as the rectangle contained by the segments of the first mentioned axis is to the square of the perpendicular.

Fig. 1. n. 1. 2.

Let H be the point, in the ellipsis, from which the straight line is drawn perpendicular to either axis, &c.

First, let HK be perpendicular to AB the greater axis; the square of AB is to the square of FG as the rectangle AKB to the square of HK. Having drawn to the foci D, E two straight lines HD, HE, place, from the vertex of the greater axis, and towards the centre C, the straight line BL equal to HE the lesser of them; and because CB, CE, CK, CL, are (4. 2.) proportionals, their squares are also proportionals: but the square of CB is (5. 2. Elem.) made up of the square of CK and the rectangle AKB; and the square of CE of the (5. 2. Elem.) square of CL and the rectangle DLE; therefore the square of CB is (4. of this and 19. 5. Elem.) to the square of CE, as the remaining rectangle AKB to the remaining rectangle DLE; and, by conversion, the square of CB is (5. 2. Elem.) to the rectangle AEB, as the rectangle AKB to its excess above the rectangle DLE, that is, as the rectangle AKB to (preceding prop.) the square of KH : but (2. 2.) the rectangle AEB is equal to the square of CF : therefore, as the square of CB to the square of CF, so is the rectangle AKB to the square of KH; the square, therefore, of AB is to that of FG, as the rectangle AKB to the square of HK.

Fig. 1. n. 1. 2. In the other case, let HP be perpendicular to the lesser axis; the square of FG is to the square of AB, as the rectangle GPF to the square of HP. The square of CB, as hath been proved, is to the square of CF, as the rectangle AKB to the square of HK or PC: consequently the square of CF (prop. B. and 19. 5. Elem.) is to the square of CB, as the rectangle FPG to the square of CK, that is, as the rectangle FPG to the square of HP.

Cor. 1. Hence the squares of straight lines drawn from points of an ellipsis perpendicular to either axis, are to one another as the rectangles contained by the segments of that axis. Fig. 2. For let HM, PQ be perpendicular to the axis AB in M and A; and, by the proposition, the rectangle AMB is to the square of HM as the square of AB to the square of FG, that is, as the rectangle AQB to the square of PQ; and, alternately, the rectangle AMB is to the rectangle AQB, as the square of HM to the square of PQ.

Fig. 2.

COR. 2. If a circle be described upon either axis as a diameter, and MH, QP perpendicular to that axis, meet the circumference in the points N, R; these perpendiculars between the axis, and the circumference of the circle, are to one another as their segments between the axis and the ellipsis: for the rectangles AMB, AQB are equal (35. 3. Elem.) to the squares of MN, QR, each to each: therefore the square of MN is to the square of QR, as the square of MH to the square of QP; consequently the straight lines MN, QR, MH, QP themselves are also (22. 6. Elem.) proportionals.

PROP. VII. THEOR.

Every straight line terminated both ways in an ellipsis, and parallel to either axis, is bisected by the other axis; that is, the axes are conjugate diameters.

Fig. 2.

Let HL be parallel to the axis FG, and meet the other axis AB in M, and the circle described upon AB in the points N, O; then, as MN to MO, (2. cor. preceding prop.) so is MH to ML: and because NO is cut at right angles by AB, MN is equal to (3. 3. Elem.) MO; therefore MH is also equal to ML.

PROP. VIII. Theor.

Every straight line terminated both ways in an ellipsis, and bisected by one axis, is parallel to the other.

Fig. 2.

If the straight line HP is bisected in S by the axis FG, it is parallel to the other axis AB. Draw HM, PQ parallel to FC, and let them meet the circle described upon AB in the points N, R: then, because HM, FC, PQ are parallels, as HS to SP, so is MC to CQ; but HS is equal to SP; therefore MC is equal to CQ; consequently MN is (14. 3. Elem.) equal to QR; and therefore HM is equal to (2. cor. 6. 2.) PQ: but HM is also parallel to PQ; therefore HP is (33. 1. Elem.) parallel to MQ.

Cor. Hence straight lines HM, PQ, parallel to either axis FG, and which cut off between the centre and the points where they terminate in the other axis, equal segments MC, QC of this other axis, are equal to each other: and, on the contrary, if HM, PQ be equal to each other, and parallel to either axis, they cut off (2. cor. 6. 2. and 14. 3. Elem.) equal segments MC, QC of the other axis.

PROP. IX. Theor.

Of all diameters the greater axis is the greatest, and the lesser axis the least; and a diameter which is nearer to the greater axis, is greater than one more remote.

Fig. 2.

Let CB be half the greater axis, and FC half the lesser; let CH be any other semidiameter, and draw HM at right angles to CB: and because the square of CB is to the square of CF as the rectangle AMB to the square of HM, and that (4. cor. 1. 2.) CB is greater than CF; therefore the rectangle AMB is greater than the square of HM: to these unequals add the common square of CM; and the square of CB (5. 2. Elem.) will be greater than that of CH (47. 1. Elem.); and consequently the straight line CB will be greater than CH. Next, draw HS at right angles to the lesser axis, and it may, in the same manner, be demonstrated, that CF is less than CH: therefore, of all semidiameters, CB is the greatest, and CF the least.

Fig. 2. Let CT be more remote from the greater axis than CH; and then CH will be greater than CT: draw TV parallel to HM, and let it meet AB in V, and the ellipsis again in X, and let HZ be parallel to MV, and make CQ equal to CM.

Because the rectangle AVB, which consists of * the rectangles AMB and QVM, is

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See fig. 2 of this book.

* This is demonstrated in prop. 31. book 7. of Pappus Alexandrinus. The proposition is to this purpose. "If in a straight line AB two equal straight lines AQ, BM be taken, and also any point V between Q and M; the rectangle AVB is equal to the rectangles AMB, QVM together." For AB being bisected in C, the rectangle AVB, together with the square of CV, is (5. 2. Elem.) equal to the square of CB, that is, to the rectangle AMB,

to the square of VT, which (5. 2. Elem.) is made up of the square VZ and rectangle TZX, as (1. cor. 6. 2.) the rectangle AMB to the square of MH or VZ; the whole the rectangle AVB, is to the whole the square of VT, as (19. 5. Elem.) the remaining rectangle QVM to the remaining rectangle TZX: but the rectangle AVB is (6. of this and prop. A. 5. Elem.) greater than the square of VT; therefore the rectangle QVM is also greater than TZX. To these unequals add the square of CV, and the square of CM (5. 2. Elem.) will be greater than the rectangle TZX together with the square of CV. Add to the same unequals, the square of MH or ZV, and the squares of CM, MH together, will be greater than the squares of VT, VC together; that is, the square of CH is greater than the square of CT; consequently CH is greater than CT.

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together with the square of CM (5. 2. Elem.) that is, to the rectangle AMB, together with the rectangle QVM, and the square of CV (5. 2. Elem.) Take away the common square of CV, and the remaining rectangle AVB is equal to both the remaining rectangles AMB, QVM.

LEMMA I.

Fig. 3. 4. If a point A be taken in a straight line AE, and two parallels BC, DE be drawn on the same side of AE, and on the same side of the point A, or on the contrary sides of AE, and on the contrary sides of the point A; if these parallels have the same ratio to each other, as the segments of AE, which are intercepted between them and the point A; the extremities B, D of the parallels and the point A are in the same straight line.

Fig. 3. In the one case, complete the parallelograms AB, AD; which are similar, and have the angle at A common: consequently they are about the same (26. 6. Elem.) diameter, that is, the points A, B, D are in the same straight line.

Fig. 4. In the other case, join BA, DA: and because BC is to CA as DE to EA, and the

angle BCA equal to DEA, the triangles ABC, ADE are equiangular (6. 6. Elem.); consequently the angle EAD is equal to the angle CAB: and therefore AB, AD make (14. 1. Elem.) one straight line.

LEMMA II.

Fig. 5. 6. 7. 8.

If two parallel straight lines AC, BD be drawn from two points A, B, and other two parallels AE, BF, from the same points, coinciding or not with the parallels AC, BD, and having the same ratio to each other as the parallels AC, BD, and lying on the same side of AB, or on the contrary sides of it, according as the parallels AC, BD are on the same side of AB or on the contrary sides of it; and if a straight line be drawn either through the extremities of the two first, or two last of these parallels, the point G where it meets AB, is in the same straight line with

BOOK II. the extremities of the other two parallels.

Let a straight line be drawn through C, D the extremities of the parallels AC, BD, and let it meet the straight line AB in the point G; the extremities, E, F of the other two parallels AE, BF, and the point G, are in the same straight line. For since the triangles AGC, BGD are equiangular, AG is to BG as AC to BD: but AC is to BD (by hypothesis) as AE to BF; therefore AG is to BG as AE to BF: consequently, by the preceding lemma, the points G, E, F are in one and the same straight line. Also, if a straight line be drawn through the extremities E, F of the parallels AE, BF the point where it meets AB, and the extremities C, D of the other two parallels, are in the same straight line.

LEMMA III.

Fig. 9. 10. 11. The same things being still supposed as in the second lemma, if through the extremities of either of the two parallels

PLATE IV.

Fig. 1. N. 1.

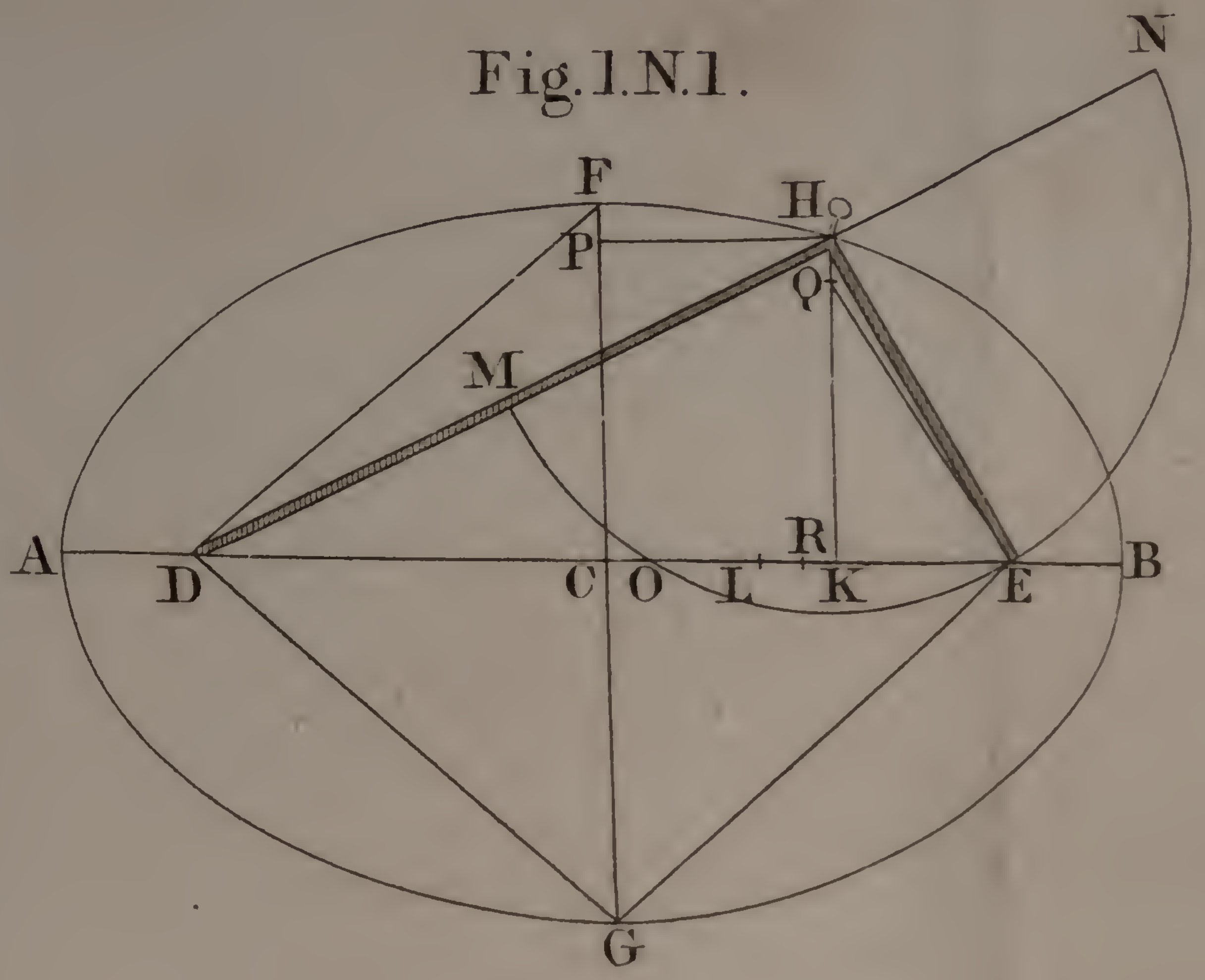


Fig. 3.

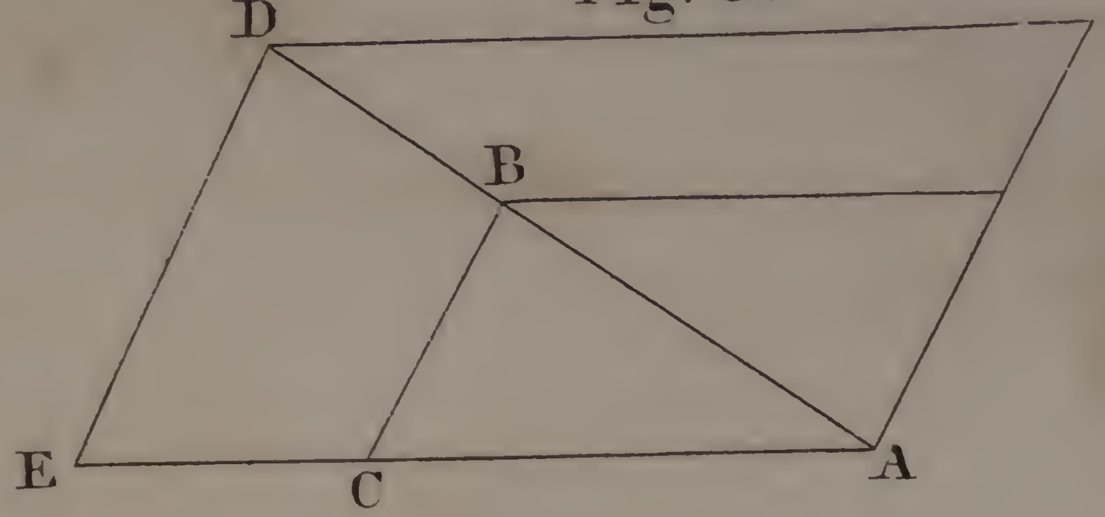


Fig. 4.

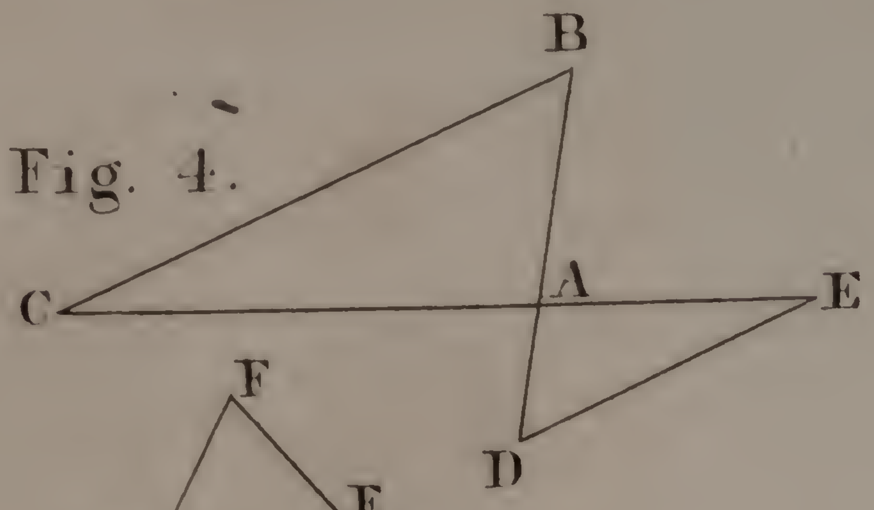


Fig. 1. N. 2.

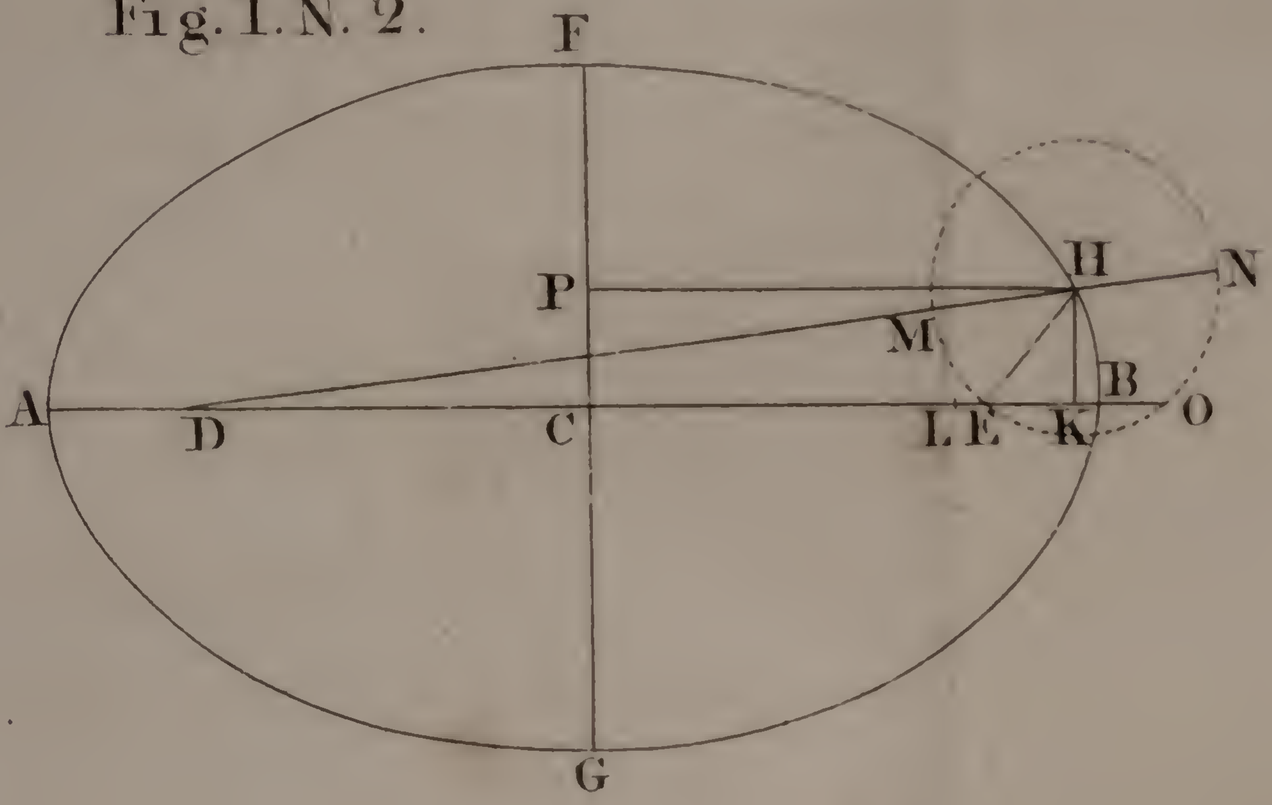


Fig. 5.

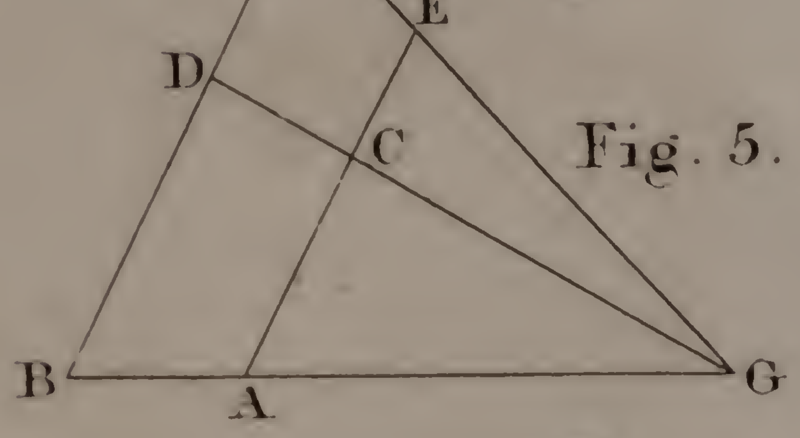


Fig. 6.

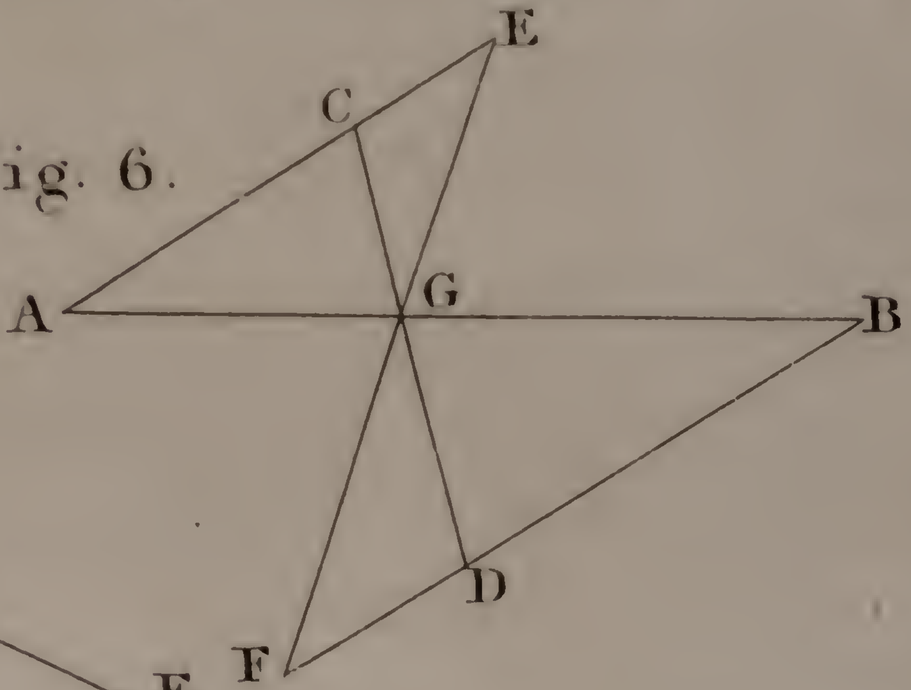


Fig. 7.

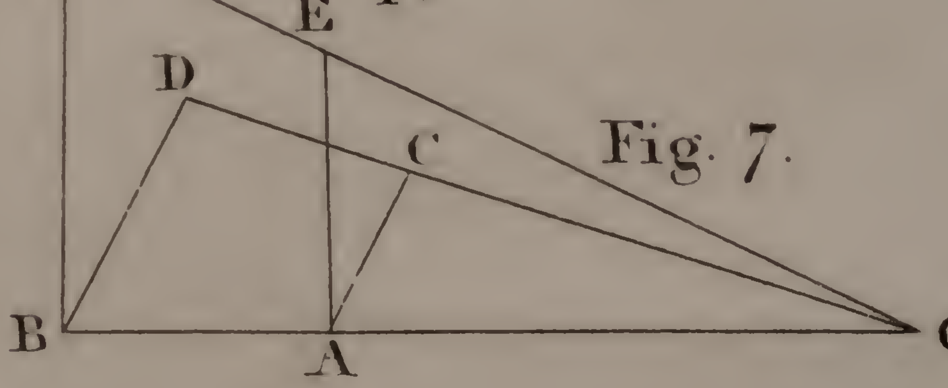


Fig. 2.

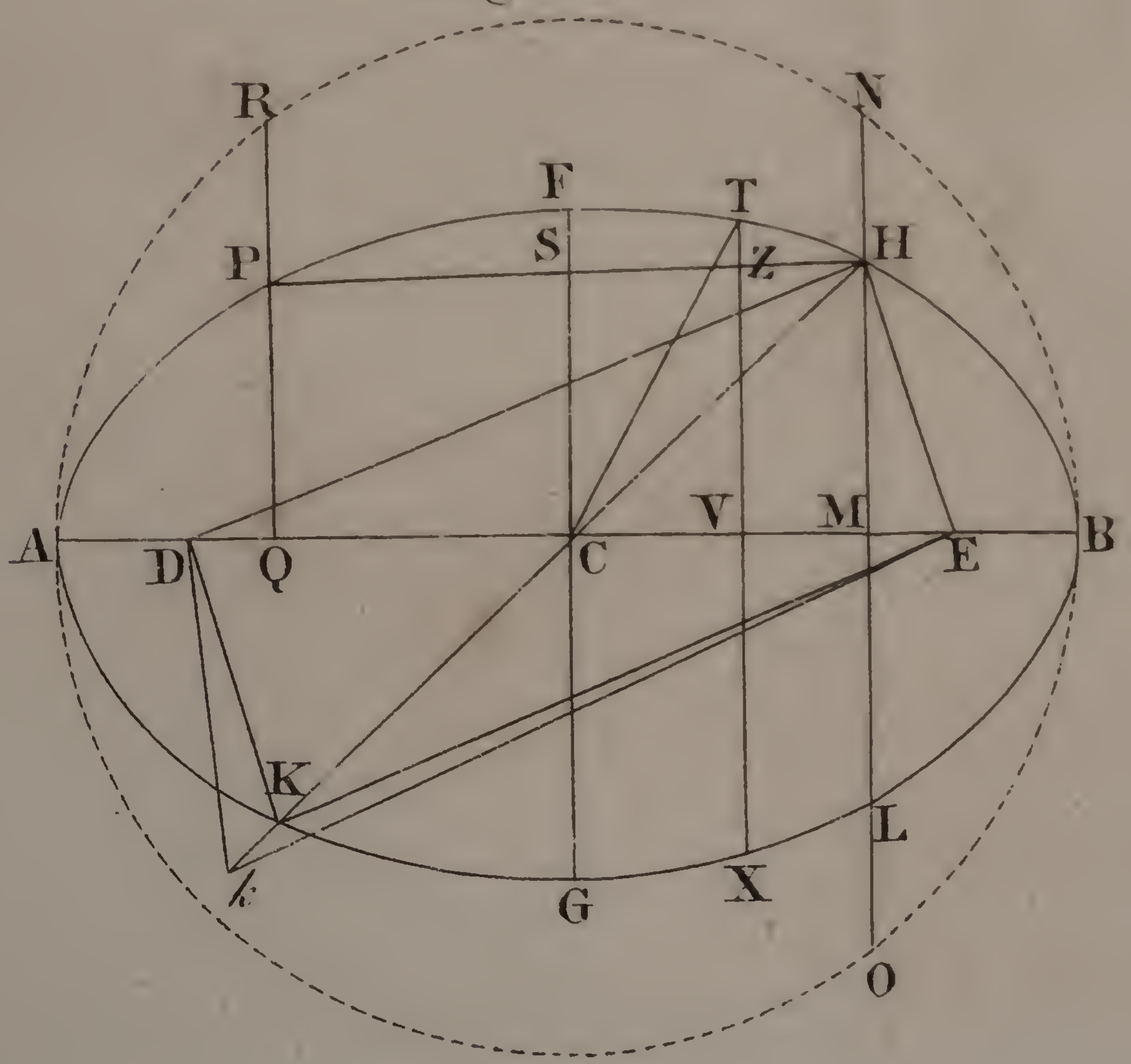
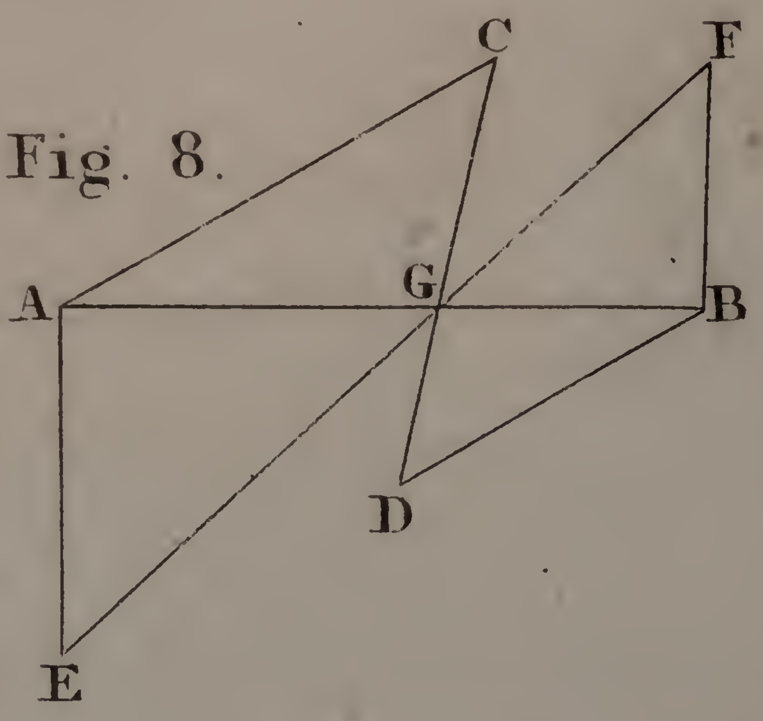


Fig. 8.



two straight lines be drawn parallel to each other, and intersecting AB; the straight lines that join the two points of intersection, and the corresponding extremities of the other two parallels, are likewise parallel.

Let CH, DK be drawn parallel to each other through the extremities C, D, of the parallels AC, BD; and let them meet AB in the points H, K; the straight lines EH, FK, which join the extremities of the other two parallels AE, BF, and the corresponding points H, K in the straight line AB, are likewise parallel.

Because both AC, BD, and CH, DK, are parallels, the triangles ACH, BDK are equiangular; therefore AH is to BK as AC to BD, that is, by hypothesis, as AE to BF; and the angle EAH is equal to FBK: therefore the triangle AHE (6. 6. Elem.) is equiangular to BKF: therefore the angle AHE is equal to BKF; and consequently EH is (27. and 28. 1. Elem.) parallel to FK.

PROP. X. THEOR.

Fig. 12. If from any point of an ellipsis E, which is not the vertex of either axis, a straight line EF be drawn parallel to either axis CD, meeting a circle described upon the other axis in the point G, and if another straight line be drawn touching the circle in the point where the parallel meets it; this other straight line meets the common diameter of the ellipsis and the circle; and a straight line drawn from the point where it meets that diameter, to the point E in the ellipsis, touches the ellipsis: also a straight line which is drawn through the vertex of either axis parallel to the other axis, touches the ellipsis.

Let AB be the other axis of the ellipsis, and C the centre of the ellipsis and circle, and join CG; let GN touch the circle in G, and meet the common diameter AB in H; the straight line that joins the point H and the point E taken in the ellipsis, touches the ellipsis.

Because CGN is a right angle, and GCB less than the right angle DCB; GN, CB will necessarily meet: let them meet in H, and join HE; HE will touch the ellipsis in the point E: if not, let it, if possible, meet the ellipsis again in K, and through K draw a straight line parallel to EF, to meet AB in L, and the circle in M: then, because EF is to KL as GF to ML, (2. cor. 6. 2.) and the points H, K, E are in one straight line; the points H, M, G are likewise in one (lemma 2.) straight line: consequently the straight line HG meets the circle in two points G and M: but, by hypothesis, the same HG touches the circle; which is absurd: therefore, HE meets the ellipsis no where but in the point E; and consequently touches it in this point.

Fig. 12.

Draw, through D, the vertex of either axis CD, a straight line parallel to the other axis AB; this straight line touches the ellipsis: for if not, it will meet the ellipsis in more points than one, let it, if possible, meet the ellipsis again in O; and through O draw a straight line parallel to CD, meeting AB in P, and the circle in Q; also let CD meet the circle in R. Then, because CO is a parallelogram, CD is equal to PO: but as CD to PO, so is CR to PQ (2. cor. 6. 2.); CR, therefore, and PQ are equal: but they are also parallel; consequently, if QR be joined, it will be parallel to CP; and the angle QRC is, therefore, a right angle. And thus RQ, drawn perpendicular to a diameter of the circle from the extremity of that diameter, falls within the circle; which is absurb (16. 3. Elem.): therefore DO touches the ellipsis.

Cor. 1. If EH touch the ellipsis, and GH be joined, it may be shown in the same manner that GH touches the circle.

Cor. 2. From a point in an ellipsis, only one straight line can be drawn which will touch that ellipsis: were it possible for two straight lines to touch the ellipsis in one and the same point, two could also touch the circle in one point; which is against the cor. to 16. 3. Elem.

Cor. 3. A straight line cannot meet an

ellipsis in more than two points; for could a straight line meet the ellipsis in more than two points, a straight line could also meet the circle in more than two points; which would be an absurdity: the ellipsis, therefore, is convex on the side where straight lines touch it, but concave on the contrary side.

Cor. 4. The angle contained by any diameter, which is not either axis of the ellipsis, and that part of the tangent at its vertex which meets the greater axis, is greater than a right angle. Let CE be the diameter, EH the tangent, and AB the greater axis, the other things remaining as in the proposition; the angle CEH is greater than CGH, that is, (21. 1. Elem.) greater than a right angle.

Cor. 5. The proposition points out a method by which, if the greater axis be given in position and magnitude, a straight line can be drawn that shall touch an ellipsis described upon the axis, in a given point.

Cor. 6. Two straight lines touching an ellipsis in the vertices of a diameter are parallel.

Fig. 12.

Let EH, ST touch the ellipsis in the vertices of the diameter ECS, and let them meet the axis AB in H, T; draw EF, SV perpendicular to the same axis, meeting the circle described upon it in G, X; join GH, XT, which, by cor. 1. will touch the circle in G, X: and because GF, XV are divided in the same ratio, (2. cor. 6. 2.) in the points E, S, and that ECS is a straight line; therefore the points G, C, X are also in a straight line (lem. 2.): and because GH, XT, which touch the circle in the vertices of the diameter GX, are parallels, EH, ST are also parallels, (lem. 3.)

PROP. XI. Theor.

If from a point of an ellipsis two straight lines be drawn to the foci, and a straight line be drawn bisecting the angle adjacent to that contained by these two straight lines; that straight line touches the ellipsis.

Fig. 13. Case 1. When the straight line bisecting the angle is parallel to the greater axis. Let AB be the greater axis, C the centre, and D, E the foci; from a point F of the ellipsis draw the straight lines FD, FE; let FH, which bisects the angle EFG, be parallel to the greater axis AB; FH touches the ellipsis. Make FG equal to FE, join EG, and let it meet FH in H; then, since FE is equal to FG, and FH common, and the angle EFH equal to GFH, EH is equal to HG: and FH, ED being parallels, EH is to HG as DF to FG; DF, therefore, is equal to FG, that is to FE; and in the triangles DFC, EFC, FC is common, and DC equal to CE; therefore (8. 1. Elem.) the angles at C are equal: and thus each of them is a right angle; therefore CF is (def. 6. 2.) the half of the lesser axis; and consequently, FH, which is parallel to the other axis, touches (10. 2.) the ellipsis.

Fig. 14.

Case 2. Let FH be inclined to the greater axis, and meet it in K; and the remaining part of the construction in the first case being repeated, draw FL at right angles to the same axis, and let it meet the circle described upon AB in the point M: joining MK, and drawing MC to the centre, make BN equal to EF; AN will consequently (1. 2.) be equal to DF. And since the outward angle EFG of the triangle DFE is divided into two equal parts by the straight line FH, which meets the base

DE in the point K; therefore DK is to KE as DF to FE (Prop. A. 6. Elem.), that is, as AN to NB: and by composition, DK together with KE is to KE as AB to NB: take the halves of the antecedents, and CK * will be to KE as CB to BN; and, by conversion and alternation, CK is to CB as CE to CN, that is, as CB to CL (4. 2.): but CB is equal to CM; therefore, as CK to CM, so is CM to CL: therefore the triangle CMK is equiangular (6. 6. Elem.) to CLM. But CLM is a right angle; therefore the angle CMK is also a right angle; and consequently the straight line MK touches the circle (16. 3. Elem.); therefore FK touches the ellipsis (10. 2.).

Otherwise: Produce DF to G, so that FG may be equal to FE; and in FK take any point O, and join OD, OE, OG, and EG, and let this last meet FK in P; then, because FE is equal to FG, FP common, and the angle EFP equal to GFP; therefore EP, GP are equal, and the angles at P right angles; therefore OE is equal to OG: but DO, OG are

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* See Prop. 8. Elem. Plane Trigon. annexed to our author's edition of Euclid's Elements.

together greater than DG; consequently DO, OE are also greater than the same DG, that is, than DF, FE together, or than the greater axis AB: the point O is, consequently, without the ellipsis (3. cor. 1. 2.); and, consequently, FK touches the ellipsis.

Cor. Conversely, if a straight line FK touch the ellipsis, and DF, FE be drawn from the point of contact to the foci; the angles DFO, EFK, which DF, FE make with the tangent in contrary directions of it, are equal. For let them be unequal; then DF being produced to G, the angles EFK, GFK are likewise unequal: but by the proposition, a straight line which bisects the angle EFG touches the ellipsis; and by the hypothesis, FK which does not bisect the angle EFG likewise touches it in the point F; which is (2. cor. 10. 2.) absurd.

PROP. XII. Prob.

The greater axis of an ellipsis being given in position and magnitude, and the foci being given, to draw a straight line parallel to another straight line given in

BOOK II. position, and which shall touch the ellipsis.

Fig. 14. Let AB be the greater axis, D, E the foci; let ST be a straight line given in position: from E, either of the foci, draw EV at right angles to ST, and from the other focus D, as a centre, with a distance equal to AB, describe a circle: this circle will meet EV in two points; of which let G be either; and having joined DG, draw EF to it, making the angle GEF, equal to EGD; and through F draw FK parallel to ST; FK touches the ellipsis in the point F.

Because the angle GEF is equal to EGF, EF is equal to FG; therefore DF, FE are together equal to DG, that is, to the greater axis AB: the point F is, therefore, in (3. cor. 1. 2.) the ellipsis. Let FK meet the straight line VE in P; and because FK, ST are parallels, and EV being at right angles to ST is likewise at right angles to FP: therefore the angles EFP, GFP are equal; and therefore (preced. prop.) FK touches the ellipsis. In like manner, by employing the other point, where the circle described from the centre D

PLATE V.

Fig. 9.

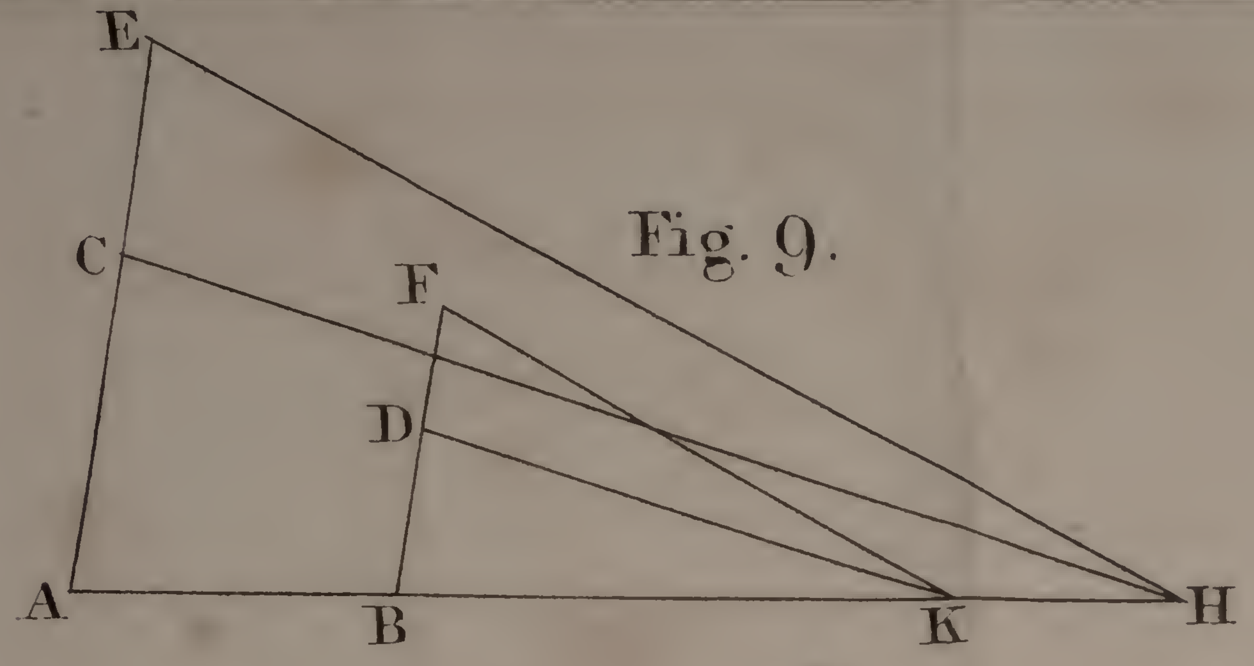


Fig. 13.

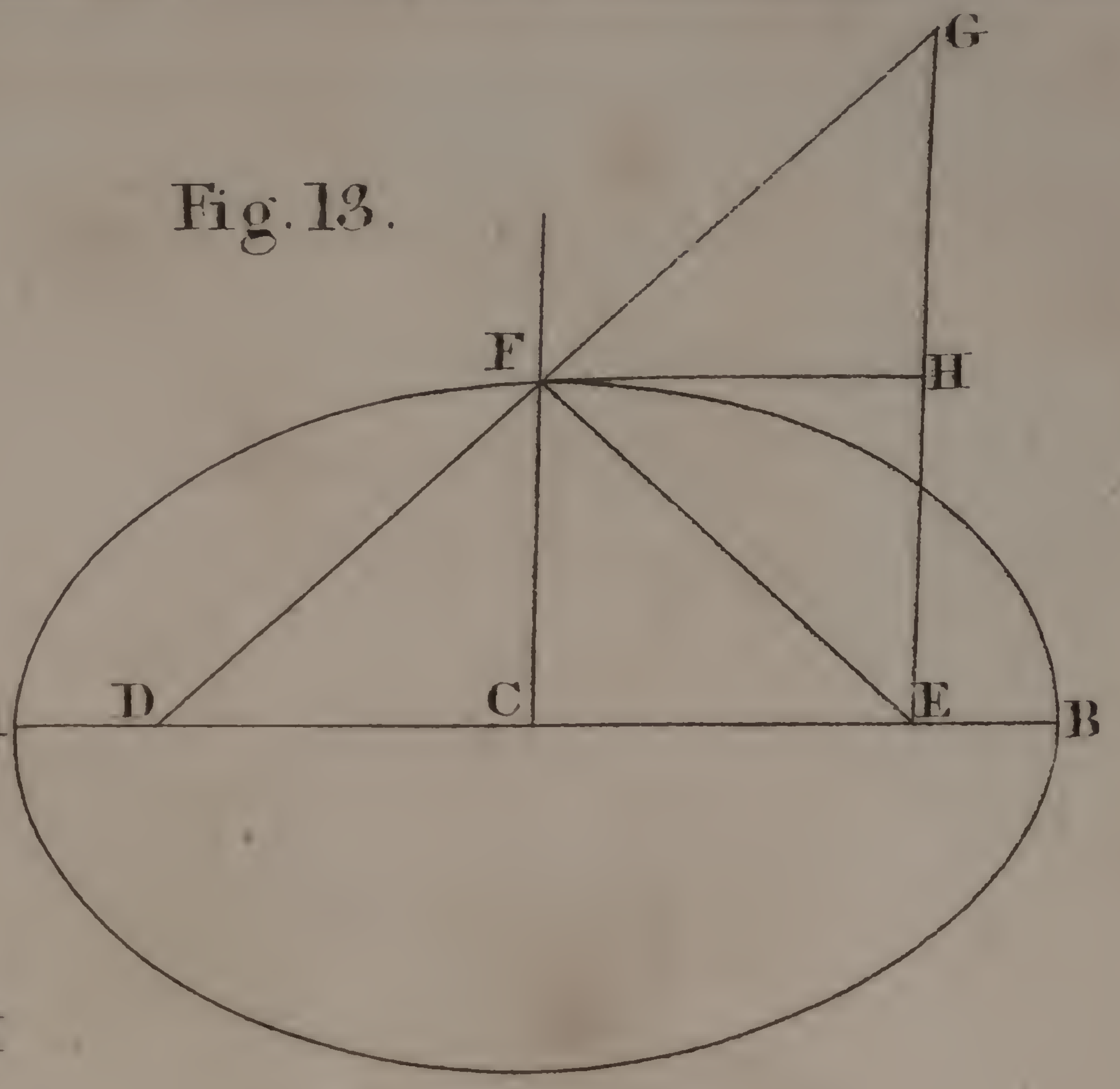


Fig. 10.

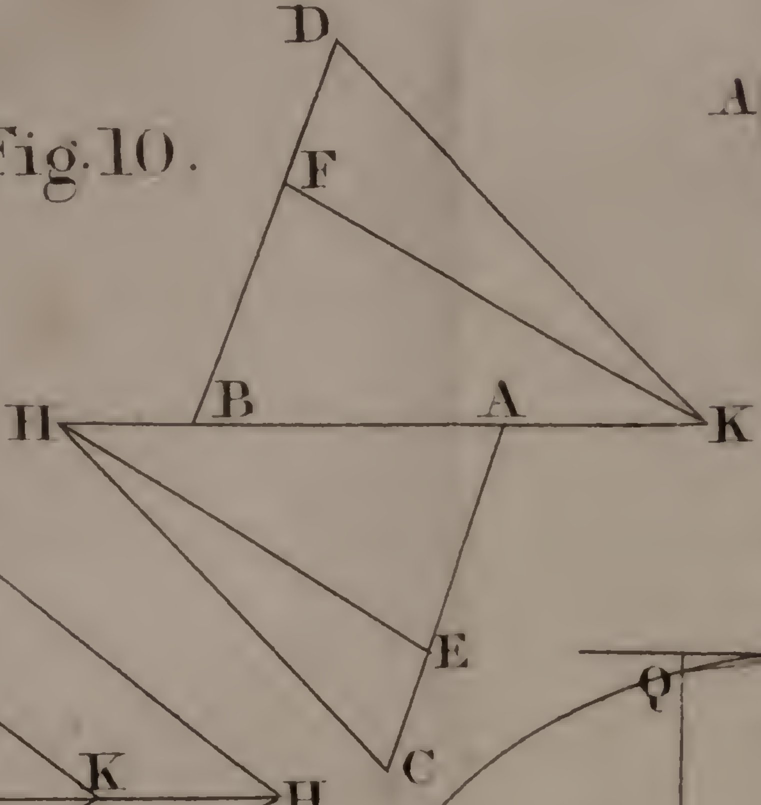


Fig. 11.

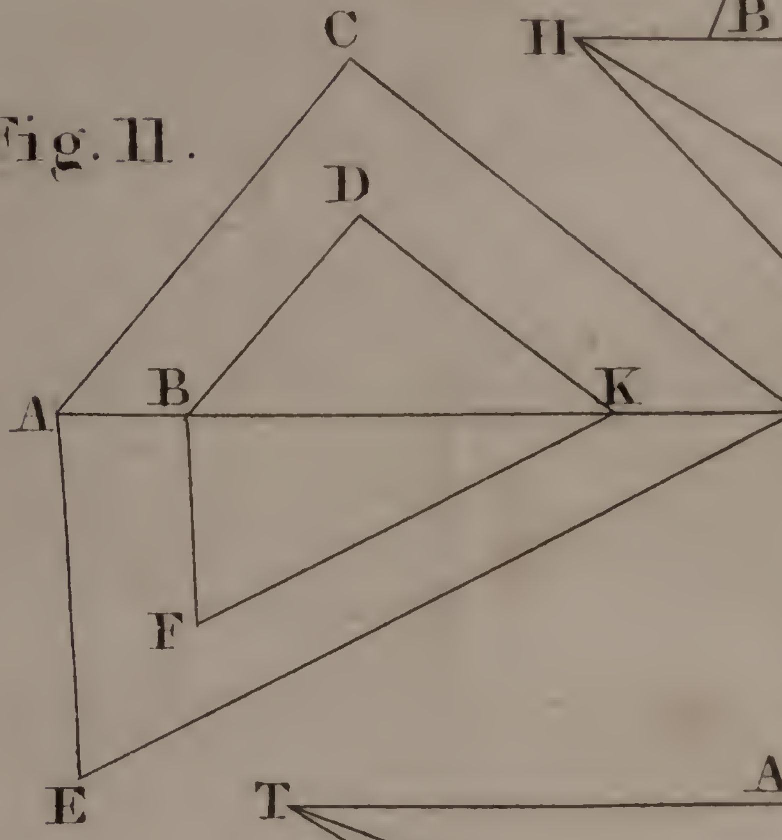


Fig. 12.

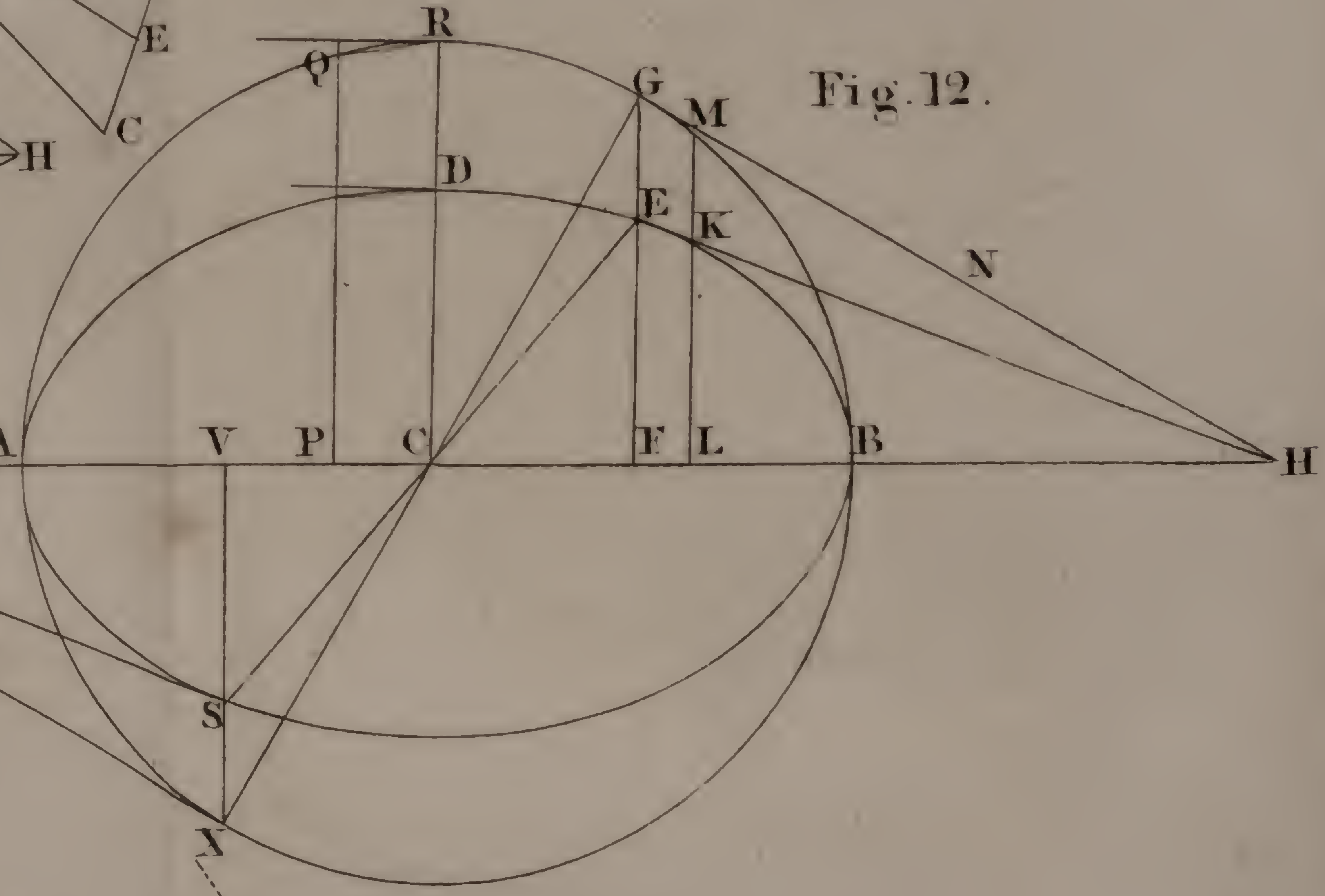
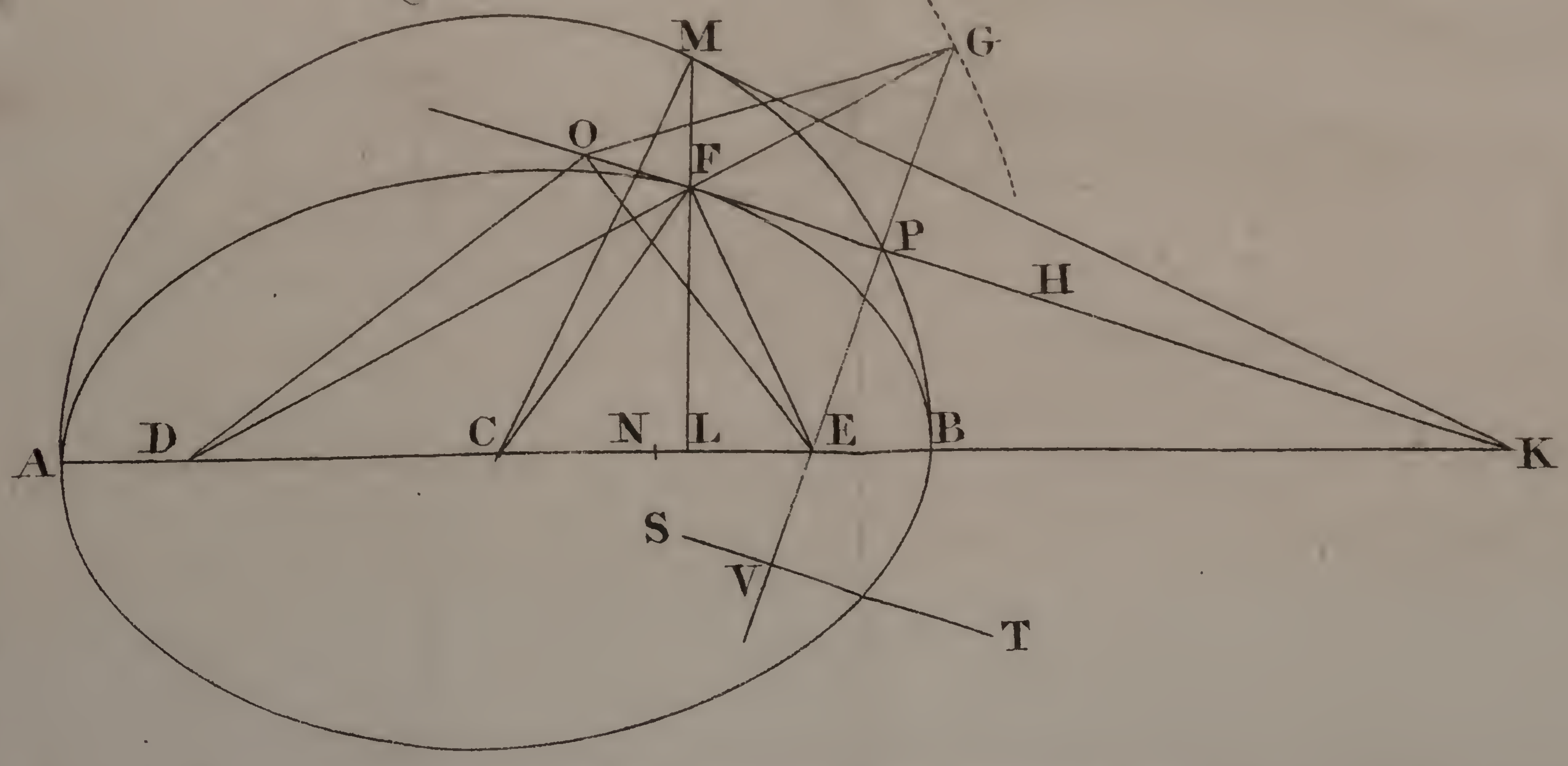


Fig. 14.



meets EV, another tangent can be drawn parallel to ST.

PROP. XIII. Theor.

Every straight line parallel to a straight line that touches the ellipsis, and terminated both ways by the ellipsis, is bisected by the diameter that passes through the point of contact.

If the diameter be either of the axes; a tangent drawn through its vertex is parallel to the other axis, (10. and 2. cor. 10. 2.); and thus a straight line parallel to the tangent is also parallel to this other axis; and consequently is bisected by the diameter that passes through the point of contact (7. 2.).

Fig. 15. n. 1. 2.

Now let CE be any other diameter, and let EF touch the ellipsis in its vertex; let GH, parallel to EF, be terminated in G, H, two points of the ellipsis, and meet the diameter CE in K; then shall GK be equal to KH.

Let the tangent EF meet either axis AB in F; let GH meet the same axis in L; through the points E, G, H draw EM, GN, HO per-

BOOK II. pendiculars to AB, and let them meet the circle described upon AB in the points P, Q, R; join FP, LQ, CP; let LQ meet CP in the point S, and join SK. Then, because EF touches the ellipsis, FP (1. cor. 10. 2.) touches the circle; and because the parallels QN, RO are cut in the same ratio in the points G, H, and that HGL is a straight line, the points L, Q, R are in a straight line (lemma 2.). Again, since the parallels PM, QN are cut in the same ratio in the points E, G, through which the parallels EF, GL, are drawn; therefore FP, LQ are likewise parallels (lem. 3.); consequently, CS is to SP, as CL to LF, that is, as CK to KE: therefore KS is (2. 6. Elem.) parallel to EP, and consequently to QN, RO; therefore GK is to KH, as QS to SR: but QS is equal to SR, because CP being at right angles to the tangent PF, is also at right angles to RQ, which is parallel to PF; therefore GK is equal to KH.

Cor. 1. Conversely: any straight line GH, terminated both ways by the ellipsis, and bisected by the diameter CE; or, in other words, any straight line ordinately applied to the diameter CE, is parallel to EF, the tangent at its vertex. If not, draw a tangent that shall be (12. 2.) parallel to GH; and GH will be bisected by the diameter which passes through the point of contact: but, by hypothesis, the same GH is bisected by another diameter CE; which is absurd.

Cor. 2. All straight lines ordinately applied to the same diameter are parallel to one another.

Cor. 3. If several parallels be terminated both ways by an ellipsis, the diameter which bisects one of them, bisects also the rest of them: for that one which is bisected by a diameter, is parallel to the tangent in the vertex of that diameter; and consequently the rest are parallel to the same tangent; and therefore are bisected by the same diameter.

Cor. 4. On the contrary: a straight line which bisects several parallels terminated both ways by an ellipsis, is a diameter. For if it be not, draw a diameter bisecting one of the parallels, and this diameter will also bisect the others: but, by hypothesis, each of them is bisected by another straight line; which is absurd. And if, from the point of contact, a straight line be drawn bisecting another straight line parallel to the tangent, and terminated both ways by the ellipsis, that straight line is a diameter. For if it be not, draw a diameter through the point of contact; this diameter, by the prop. also bisects the parallel to the tangent; which is absurd.

Cor. 5. A straight line which is drawn through the vertex of a diameter, and is parallel to an ordinate to that diameter, touches the ellipsis.

Cor. 6. Two straight lines in an ellipsis, which pass not through the centre, do not bisect each other: for if they bisected each other, they would be each of them parallel to the tangent in the vertex of the diameter drawn through the point of bisection, and of consequence they would be parallel to each other; which is absurd.

PROP. XIV. Theor.

Two diameters of an ellipsis, either of which is parallel to a straight line touching the ellipsis in either vertex of the other, are conjugate diameters.

Fig. 15. n. 1. 2.

Let ET, VX be two diameters; let either of them, VX, be parallel to EF, a straight line touching the ellipsis in the vertex E of the other; then ET, VX are conjugate diameters.

Through the vertices E, V draw perpendiculars EM, VY to either axis AB, meeting the circle described upon it in the points P, Z, on the same sides of the diameter AB with the points E, V; and from the point F, where EF meets AB, draw the straight line FP, which will touch the (1. cor. 10. 2.) circle in P; and from the point Z, the straight line $Z\alpha$ touching the circle, and meeting the axis in α, join αV; and αV will touch the (10. 2.) ellipsis; and from the centre draw CP, CZ.

Then, because the parallels PM, ZY are cut in the same ratio in the points E, V through which are drawn the parallels EF, VC; therefore FP, CZ are (lem. 3.) also parallels: but CPF is a right angle; consequently PCZ is also a right angle: and $CZ\alpha$ is a right angle; therefore CP, $Z\alpha$ are parallels; and consequently CE and αV are parallels: therefore straight lines parallel to CE will likewise be parallel to $V\alpha$, which touches the ellipsis in V; and of consequence they will be bisected (13. 2.) by the diameter CV: and because CV, by hypothesis, is parallel to EF, all straight lines parallel to CV will likewise be parallel to EF, and will consequently be bisected by the diameter CE; therefore CE, CV are conjugate diameters (7. def. 2.).

Cor. 1. And since CV is the only diameter that can bisect straight lines parallel to CE, and terminated both ways by the ellipsis, CV alone is the conjugate to CE.

Cor. 2. If, therefore, CE, CV be conjugate diameters, each of them is parallel to the tangent drawn through the vertex of the other.

Cor. 3. On the other hand, a straight line drawn through the vertex of a diameter, parallel to the conjugate diameter, touches the ellipsis in that vertex.

Cor. 4. A straight line parallel to a diameter, and terminated both ways by the ellipsis, is bisected by the conjugate diameter: for it is parallel to the straight line which touches the ellipsis in the vertex of this diameter; and consequently it is bisected by this same conjugate diameter. On the other hand, a straight line bisected by a diameter, is parallel to the conjugate diameter.

PROP. XV. THEOR.

If from a point in an ellipsis to either of two conjugate diameters, a straight line be drawn parallel to the other, the square of the diameter it meets, is to the square of the other diameter, as the rectangle contained by the segments into which the straight line divides the first, is to the square of that straight line.

Fig. 15.
n. 1. 2.

Let ET, VX be conjugate diameters, and from a point G of the ellipsis draw GK parallel to the diameter VX, meeting the other ET in K; then the square of ET is to the square of VX, as the rectangle EKT to the square of GK.

For since ET, VX are conjugate diameters, VX is parallel to the tangent EF, drawn through the vertex E of ET: and the same construction as in the two foregoing propositions still remaining, and what was there demonstrated being still kept in view, let SK, when produced, meet AB in the point β; and through G draw the straight line Gγ parallel to the same AB, and meeting Sβ in γ; then, because PCZ is a right angle, and that the angles CPM, MCP are together equal to a right angle; the angle PCZ is equal to the same CPM and MCP together: take away the common angle MCP, and the remaining angle MCZ is equal to the remaining CPM; and CP, CZ are equal; therefore the right-angled triangles PMC, CYZ are equal; therefore PM is equal to CY. But since PM, Sβ, and PF, SL are parallels, the triangles PFM, SLβ are equiangular: and thus CPM, SLβ are (8. 6. Elem.) also equiangular; therefore CP is to PM as SL to Lβ, that is, as SQ to Nβ (2. 6. Elem.); alternately CP is to SQ as PM to Nβ: but PM having been proved equal to CY, and that Gγ is equal to Nβ; therefore PM is to Nβ, as CY to Gγ: and the triangles CVY, GKγ being equiangular, CY is to Gγ, as CV to GK; therefore (ex æquali) CP is to SQ as CV to GK: hence the square of CP is to the square of SQ, as the square of CV to the square of GK: but the square of CP is to the square of CS as the square of CE to the square of CK; and, by conversion, and prop. 47. 1. Elem. the square of CP is to the square of SQ as the square of CE to the rectangle EKT: the square, therefore, of CE is to the rectangle EKT, as the square of CV to that of GK; alternately, the square of CE is to the square of CV as the rectangle EKT to the square of GK: therefore (15. 5. Elem.) the square of ET is to that of VX, as the rectangle EKT to the square of GK.

Universally: the square of any diameter is to the square of its conjugate, as the rectangle contained by the segments, intercepted between its vertices and a straight line ordinately applied to it, is to the square of the segment of the same straight line between the ellipsis and that diameter: for an ordinate to a diameter is parallel to the tangent drawn through the vertex of that diameter; and therefore is parallel to the conjugate diameter.

Cor. 1. The squares of straight lines ordinately applied to the same diameter, are to one another as the rectangles contained by the segments of that diameter, as was demonstrated (1. cor. 6. 2.) with regard to the axes.

Cor. 2. If ET, VX be conjugate diameters of an ellipsis AT, and if from a point G a straight line GK be drawn parallel to VX, one of the diameters, and meeting the other ET in K; and if the square of ET be to the square of VX, as the rectangle EKT to the square of GK; the point G is in the ellipsis. For if the point G is not in the ellipsis, then GK will meet it in some other point on that side of the diameter ET, on which G is; let it, if possible, meet the ellipsis in δ: then, by the proposition, the rectangle EKT is to the square of δK, as the square of ET to the square of VX, that is, by hypothesis, as the rectangle EKT to the square of GK: hence the square of δK is equal to the square of GK; and thus the straight line δK is equal to the straight line GK; which is impossible.

Cor. 3. If from two points G, ε, one of which, ε, is in the ellipsis, there be drawn to the diameter ET straight lines GK, εζ parallel to straight lines ordinately applied to the same ET; if the rectangles EKT, EζT, contained by the segments of the diameter ET, which are intercepted between its vertices, and the parallels drawn to it, have the same ratio to each other as the squares of GK, εζ; then the other point G is likewise in the ellipsis. This is demonstrated from cor. 1. in the same manner as the second corollary from the proposition.

Fig. 16.

Cor. 4. If a circle be described upon AB, a diameter of the ellipsis, and if straight lines DE, FG be drawn ordinately applied to the diameter AB; and if from the points D, F straight lines DH, FK be drawn perpendicular to the same AB, and meeting the circle in the points H, K; then the perpendiculars DH, FK shall have the same ratio to each other which the ordinates DE, FG have. For the squares of DE, FG have the same ratio to each other which the rectangles ADB, AFB have, that is, which the squares (35. 3. Elem.) of DH, FK have: therefore DH is to FK, as DE to FG, (22. 6. Elem.)

Cor. 5. And if two straight lines ordinately applied to a diameter, cut off, between the centre and the points where they meet that diameter, equal segments of it, they are equal: and if equal, they cut off, between the centre and these points, equal segments.

PROP. XVI. Theor.

Fig. 16. If from a point E of an ellipsis, a straight line ED be ordinately applied to a diameter AB, and if DH be drawn at right angles to that diameter, meeting a circle described upon the same diameter in H; and if the straight line touching the circle in H, meet that diameter in L, and if EL be drawn joining the points L and E, the line EL will touch the ellipsis in E: and conversely.

For if EL do not touch the ellipsis it will cut it; let it, if possible, meet it in another point M; and through M to the diameter AB draw MN parallel to ED; and through N draw NO at right angles to the diameter AB, meeting the circle in O on the same side of AB with H: since the parallels DE, NM are drawn from the points D, N, as also the parallels DH, NO, which have the same ratio to each other (4. cor. preced.) which DE, NM have, and that the points E, M, L are in a straight line; therefore the points H, O, L (lem. 2.) are likewise in a straight line; the straight line LH, therefore, cuts the circle: but, according to the hypothesis, it touches the circle; which is absurd: therefore LE also touches the ellipsis.

And conversely: it may be shown, after the same manner, that if EL touches the ellipsis, LH likewise touches the circle.

Cor. Hence it is manifest in what manner, if a diameter of an ellipsis be given in position and magnitude, and the angle which that diameter makes with any straight line ordinately applied to it, be given, a straight line can be drawn that will touch the ellipsis in a given point.

PROP. XVII. THEOR.

Fig. 16. If from a point E of an ellipsis a straight line be drawn touching that ellipsis, and meeting the diameter AB in L; and if from the point of contact a straight line ED be ordinately applied to that diameter; the semidiameter CB is a mean proportional between CL and CD, the segments of the diameter, intercepted, the one between the centre and the tangent, and the other between the centre and the ordinate; and the segments of the same diameter intercepted between its vertices and the tangent, have the same ratio to each other as the segments between its vertices and the ordinate.

Having described a circle upon the diameter AB, and drawn from the point D a straight line DH at right angles to AB, meeting the circle in H, join HL: then, because LH touches the (16. 2.) circle, and that HD is perpendicular to the diameter, CD, CB, CL are proportionals (8. 6. Elem.). Which is the first case.

Secondly, because CL is to CB as CB to CD, by conversion CL is to BL, as CB to BD; double the antecedents, and twice CL is to BL, as AB to BD; and, by division, AL is to BL, as AD to BD.

Cor. 1. Hence the rectangle contained by the segments of the diameter intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to that contained by the segments between the ordinate and the vertices of the diameter. For as CD, CB, CL are proportionals, the square of CB is equal to the rectangle DCL; but the square of CB is equal to the rectangle ADB, together with the square of CD (5. 2. Elem.); and the rectangle DCL is equal to the rectangle CDL, together with the same square of CD (3. 2. Elem.): take away the common square of CD, and there remains the rectangle ADB equal to CDL.

BOOK II. Cor. 2. And the rectangle contained by the segments of the diameter intercepted between the tangent and the centre, and between the tangent and the ordinate, is equal to that contained by the segments between the tangent and the vertices of the same diameter: for the rectangle ALB and square of CB are together equal (6. 2. Elem.) to the square of CL; and the rectangles LCD, CLD are together equal (2. 2. Elem.) to the same square of CL: therefore, because the square of CB has been proved equal to the rectangle DCL, there remains the rectangle ALB equal to the rectangle CLD.

PROP. XVIII. Theor.

Fig. 16. From a point E of an ellipsis let a straight line ED be ordinately applied to a diameter AB, and from the same point let a straight line EL be drawn meeting that diameter in L: if the segment (of the diameter) intercepted between the centre C and the point L, the semidiameter CB, and the segment DC between

the centre and the ordinate be proportionals, the straight line EL will touch the ellipsis in E: or secondly, if the segments between the point L, and the vertices of the diameter, and between the ordinate and those vertices, be proportionals; or if, thirdly, the four segments between the ordinate and each of the points L, A, B, C be proportionals; or, last of all, if the four segments between the point L, and each of these points A, C, D, B be proportionals; then the straight line EL touches the ellipsis.

Case 1. If EL does not touch the ellipsis, let EP touch it: therefore, by the preceding proposition, CP, CB, CD are proportionals: but, by hypothesis, CL, CB, CD are likewise proportionals; CP is therefore equal to CL; which is absurd: consequently EL touches the ellipsis.

Case 2. Because, by hypothesis, AL is to BL as AD to BD; by composition, AL and

BL together are to BL, as AB to BD: take the halves of the antecedents, and CL is to BL, as CB to BD; and, by conversion, CL is to CB, as (preced. prop.) CB to CD: therefore, by the first case, EL touches the ellipsis.

Case 3. Since DL is to DA, as DB to DC; by composition, LA is to AD, as BC or CA to CD; the remainder CL (19. 5. Elem.) is, therefore, to the remainder AC or CB, as CB to CD; and therefore, by the first case again, EL touches the ellipsis.

Case 4. Because, by hypothesis, AL is to CL as DL to BL; therefore, by division, AC or CB is to CL, as DB to BL; therefore CB is to CL, as the remainder CD to the remainder CB; and, inversely, CL is to CB as CB to CD: therefore EL touches the ellipsis, by the same first case.

PROP. XIX. THEOR.

If from two vertices of two conjugate diameters two straight lines be ordinately applied to another diameter, the square of the segment (of that other diameter) intercepted between either ordinate and the centre, is equal to the rectangle contained by the segments between the other ordinate and the vertices of that same diameter.

Fig. 17.

Let CA, CB be the two conjugate diameters, of which the points A, B are vertices; from A, B let AF, BG be ordinately applied to another diameter DE; the square of CG, intercepted between the ordinate BG and the centre, is equal to the rectangle EFD contained by the segments intercepted between the other ordinate AF and the vertices of DE: and likewise the square of CF is equal to the rectangle EGD.

Draw AH, BK touching the ellipsis in A, B, and meeting the diameter ED in H, K: then, because both CB, AH, and BG, AF, are parallel, the triangle CBG is equiangular to HAF; and because BK is parallel to CA, the triangle CBK is also equiangular to HAC; therefore CG is to FH, as CB to AH, that is, as CK to CH: but CD is a mean proportional both between CG, CK, and between CF (16. 2.) CH; therefore CF is to CG, as

CK to CH: and as hath been shown, CG is to FH, as CK to CH; therefore (ex æquali) as CF to CG, so is CG to FH: and consequently, the square of CG is equal to the rectangle CFH. But the rectangle EFD is equal to the same (1. cor. 17. 2.) CFH; hence the square of CG is equal to the rectangle EFD; take these equals from the square of CD, and there will remain the rectangle EGD equal to the square of CF (4. and 5. 2. Elem.).

COR. 1. Hence the semidiameter CD, to which the ordinates are drawn, is to its semidiameter conjugate CL, as the distance between either ordinate and the centre is to the other ordinate: for the square of CD is to the square of CL as the rectangle EFD to the square of AF, that is (by the proposition), as the square of CG to the square of AF; therefore CD is to CL, as CG to AF. In like manner, it may be shown that CD is to CL, as CL to BG.

COR. 2. The squares of the segments of the diameter to which the ordinates are drawn, between the ordinates and the centre, are to-

PLATE VI.

Fig.15.N.1.

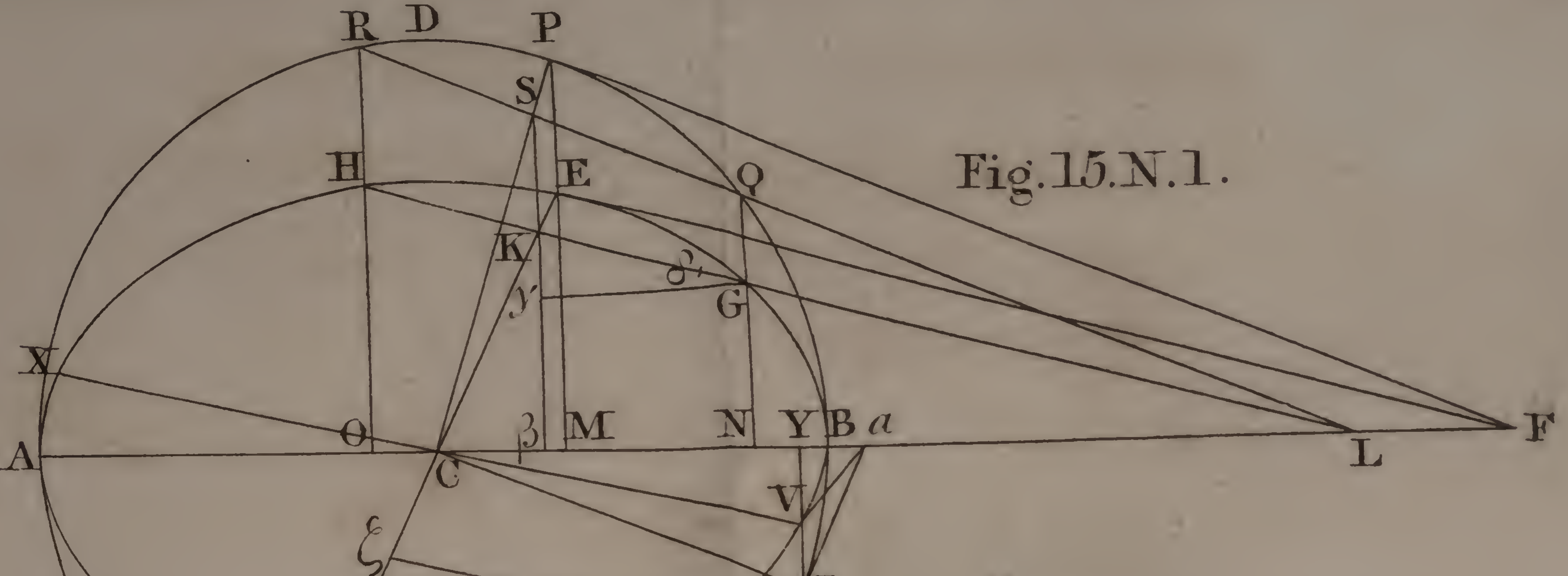


Fig.15.N.2.

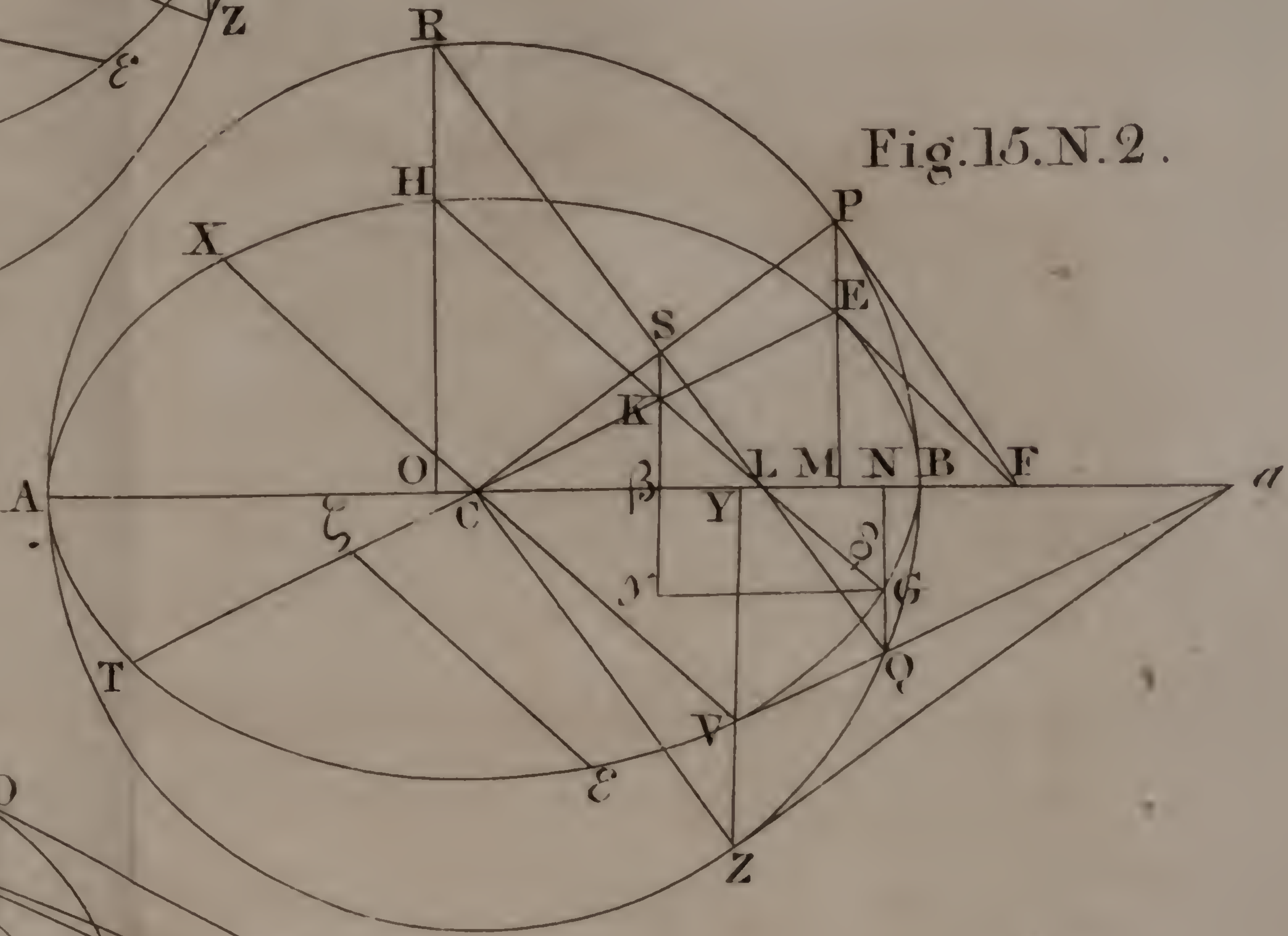


Fig.16.

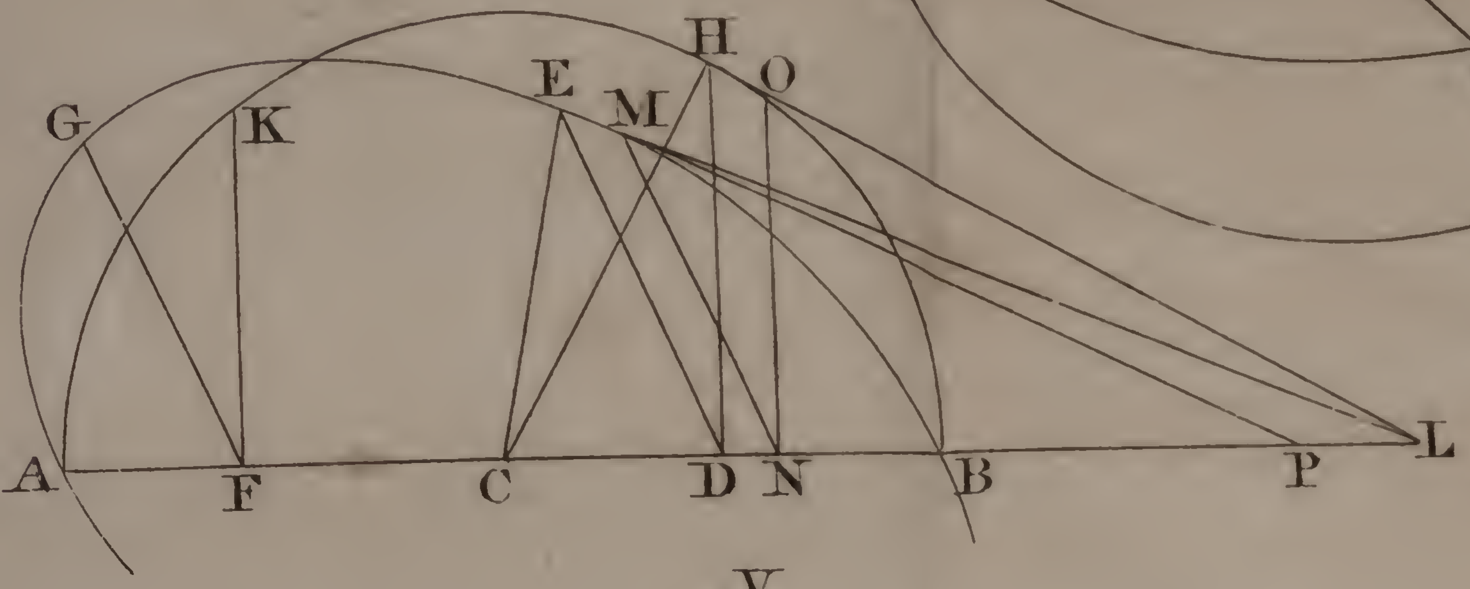
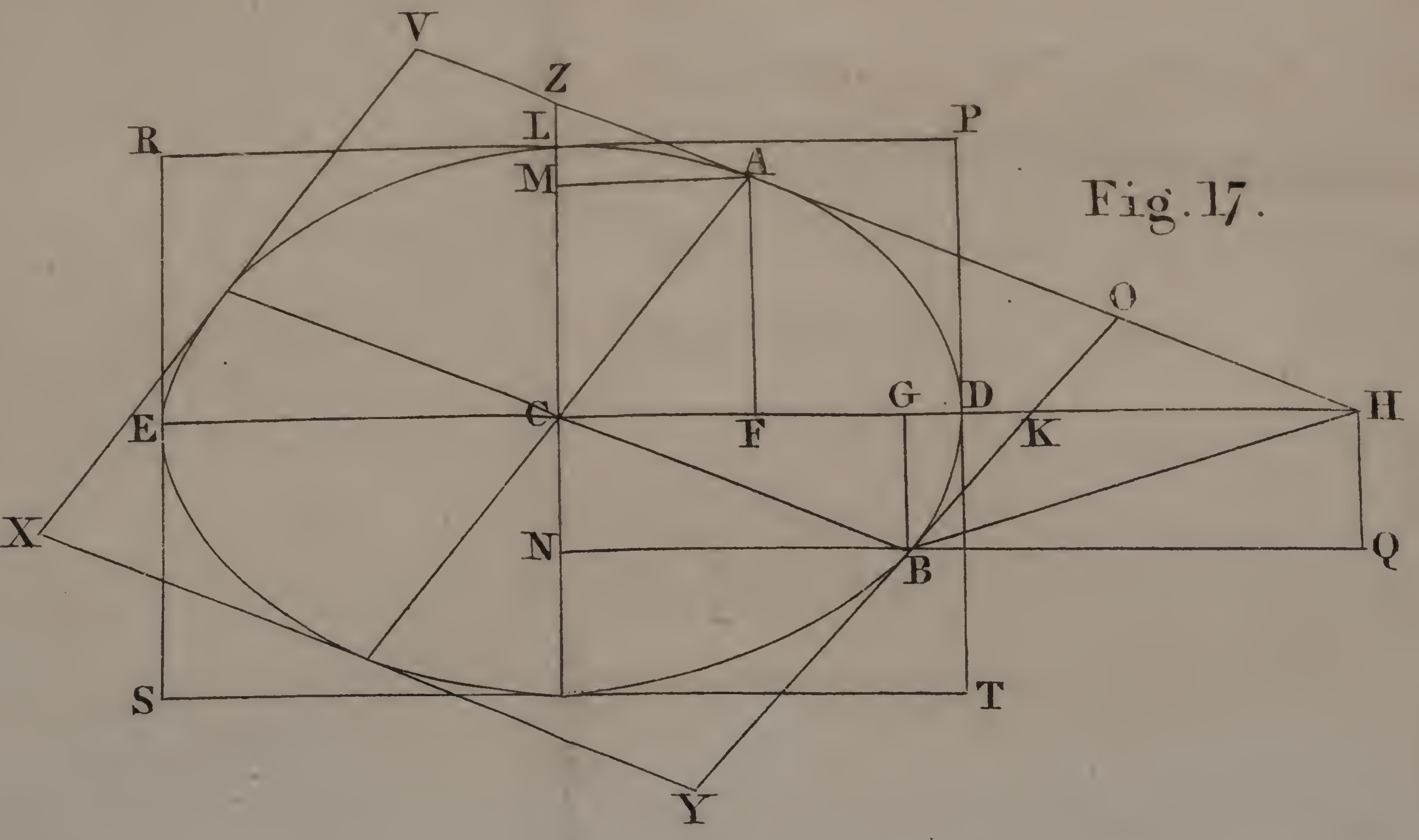


Fig.17.



gether equal to the square of the semidiameter. For since the square of CG is equal to the rectangle EFD; therefore the square of CF, together with the square of CG, is equal to the square of CF, together with the rectangle EFD, that is, to the (5. 2. Elem.) square of CD.

Cor. 3. Hence the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axis. Let CD, CL be the semiaxes, and CA, CB conjugate semidiameters; let AF, BG be perpendiculars to CD, and AM, BN perpendiculars to CL: then, because the square of CD, as was proved in the preceding corollary, is equal to the square of CF, together with the square of CG; and that, by the same corollary, the square of CL is equal to the square of CM, together with that of CN, that is, to the square of AF, together with that of BG; therefore the squares of CD, CL are together equal to the squares of CF, CG, AF, BG: but the squares of AC, BC are together (47. 1. Elem.) equal to the same squares of CF, CG, AF, BG; and therefore the sum of the squares of AC, BC is equal to the sum of the squares of CD, CL.

PROP. XX. THEOR.

If through the vertices of two conjugate diameters four straight lines be drawn touching the ellipsis; the parallelogram contained by these straight lines, is equal to that contained by the tangents drawn through the vertices of any other other two conjugate diameters.

Fig. 17.

Let the straight lines OV, OY, YX, XV touch the ellipsis in the vertices A, B, and in their opposite vertices of two conjugate diameters AC, BC; in like manner, let PT, PR, RS, ST touch the ellipsis in the vertices of the conjugate diameters DC, LC; the figures OVXY, PRST are (6. cor. 10. 2.) parallelograms, and equal to each other.

To the diameter CD draw AF, BG parallel to CL; and to the diameter CL draw AM, BN parallel to CD; and let AO, BO meet the same CD in the points H, K; and having joined BH, complete the parallelogram HCNQ:

Then, because AH touches the ellipsis, and that AF is drawn ordinately applied to the diameter CD; therefore CH is to CD, as CD to CF (17. 2.): and, by the first corollary of the foregoing proposition, CD is to CL, as CF to BG: therefore, *ex æquo*, CH is to CL, as CD to BG, or QH; and the angles DCL, CHQ are equal, for they are alternate; therefore the parallelogram DL is equal to the parallelogram (14. 6. Elem.) NH: but NH is the double of the triangle CBH, upon the same base CH, and between the same parallels; and likewise the parallelogram ACBO is the double of the same triangle CBH, upon the same base CB, and between the same parallels CB, AH: therefore ACBO is equal to NH; and the parallelogram DL, as hath been shown, is equal to the same NH; therefore the parallelograms DL, AB are equal: and thus the parallelograms PRST, OVXY, which are the quadruples of DL, AB, are likewise equal.

PROP. XXI. THEOR.

If a straight line touching an ellipsis, meet two conjugate diameters; the rectangle contained by its segments, between the point of contact and the diameters, is equal to the square of the semidiameters conjugate to that which passes through the point of contact.

Fig. 17. Let the straight line HZ touch the ellipsis in the point A, and let it meet the conjugate diameters CD, CL in H, Z; let the semidiameter CB be conjugate to CA; then the rectangle HAZ is equal to the square of CB.

For draw AF, BG parallel to the diameter CL: and since HF is to FC, as HA to AZ, the rectangles (def. 1. 6. Elem.) HFC, HAZ are similar; and HF is to HA, as CG is to CB; therefore, since the rectangles HFC, HAZ are similar, of which HF, HA are homologous sides, and that the squares of CG, CB are similar figures; the rectangle HFC (22. 6. Elem.) is to the rectangle HAZ as the square of CG to the square of CB: but the rectangle HFC is equal to the rectangle (1. cor. 17. 2.) EFD, that is, to the (19. 2.) square of CG; and therefore (14. 5. Elem.) the rectangle HAZ is also equal to the square of CB.

Cor. Hence it follows, if a straight line HAZ touch an ellipsis, and meet two diameters CH, CZ; if the rectangle HAZ be equal to the square of CB the semiconjugate to CA, which passes through the point of contact; then CH, CZ are two conjugate diameters.

PROP. XXII. Theor.

If from a point of an ellipsis, a straight line be ordinately applied to a diameter; the rectangle contained by the segments of the diameter is to the square of the ordinate, as the diameter is to its *latus rectum*.

Let F be a point in an ellipsis; from F draw FG, ordinately to the diameter AB. Fig. 18.

The rectangle AGB is to the square of FG as the diameter AB to its *latus rectum*.

For let BH be equal to the *latus rectum*; and since the diameter AB, its conjugate DE, and *latus rectum* BH (9. def. 2.), are proportionals, AB is to BH (2. cor. 20. 6. Elem.) as the square of AB to the square of DE, that is, as the rectangle AGB to the square of FG (15. 2.).

PROP. XXIII. THEOR.

If from a point of an ellipsis, a straight line be ordinately applied to a diameter, and from the vertex of the diameter a straight line be drawn at right angles to it, and equal to its *latus rectum*; the square of the ordinate is equal to the rectangle applied to the *latus rectum*; having for its breadth the abscissa between the ordinate and the vertex of the diameter, but deficient by a figure similar, and similarly situated, to the figure contained by the diameter and the *latus rectum.*

Let F be a point in the ellipsis; from F draw FG, an ordinate to the diameter AB; and from the vertex of AB draw a perpendicular BH, equal to the *latus rectum* of AB; then the square of FG is equal to the rectangle applied to BH; having the abscissa BG for its breadth, but deficient by a figure similar, and similarly situated, to the rectangle BN, contained by the diameter AB and *latus rectum* BH. Fig. 18.

Having joined AH, and from G drawn GK parallel to BH, and meeting AH in K, complete the parallelograms KLHM, ABHN: then, because the rectangle AGB is to the square of FG, as (22. 2.) AB to BH, that is, as AG to GK, that is, as the (1. 6. Elem.) rectangle AGB to the rectangle KGB; therefore (ex æquali) the rectangle AGB is to the square of FG, as the same AGB to the rectangle KGB: and thus the square of FG is equal to the rectangle KGB, applied to the *latus rectum* BH; which rectangle has for its breadth the abscissa GB, but is less than the rectangle BM, by the figure KLHM, similar, and similarly situated, to BN (24. 6. Elem.).

From the square of the ordinate being thus equal to the *deficient* rectangle; or that under the abscissa and only a *part* of the *latus rectum*, Apollonius called this curve line the *Ellipsis.*

COR. If from the vertex B of the diameter AB any straight line BO be drawn equal to the *latus rectum*, though not at right angles to AB; join AO, and through G draw GP parallel to BO; the rectangle PGB is equal to the square of FG. For AB is to BH, as BG to GK, that is, as BO to GP: but BO is equal to BH, therefore GP is also equal to GK.

PROP. XXIV. PROB.

Two unequal straight lines which bisect each other at right angles, being given in position and magnitude, to describe an ellipsis of which these straight lines may be the axes.

BOOK II. Fig. 19.

Let two unequal straight lines AB, DE, which bisect each other at right angles in the point C, be given in position and magnitude; it is required to describe an ellipsis which may have AB and DE for the axes.

From the extremity D of DE, the less of the two, place DF equal to CB, the half of AB the greater; and from the centre D, with the distance DF, describe a circle which will meet AB in two points G, H; in which fix the ends of a string of the same length with the straight line AB, and describe an ellipsis, as was directed in the first definition of this book; the straight lines AB, DE will be the axes of this ellipsis.

For since the points G, H are the foci of the ellipsis, and that GC is (3. 3. Elem.) equal to CH, the point C is its centre (3. def. 2.); and since CA or CB is equal to the length of the half of the string, the ellipsis passes through the points A, B: it passes likewise through D (4. cor. 1. 2.), because GD is equal to CA; and through E, because CD is equal to CE.

PROP. XXV. Prob.

A straight line being given in position and magnitude, and a point without it being given; to describe an ellipsis, of which that straight line shall be one of the axes, and which shall pass through that given point; but the given point must be so situated, that a perpendicular drawn from it may fall between the extremities of the given straight line.

Fig. 19. Let AB be the straight line given in position and magnitude, and K the given point, so situated, that a perpendicular drawn from it towards AB may fall between the extremities A, B; it is required to describe an ellipsis which may have AB for one of the axes, and which may pass through K.

Draw KL at right angles to AB, and find a straight line DE such *, that the square of

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* Find a mean proportional X between AL and LB (13. 6.

AB may be to the square of DE, as the rectangle ALB to the square of KL; and place DE perpendicular to AB, so that they mutually bisect each other: and by the last prop. with the same AB, DE, as the axes, describe an ellipsis; this ellipsis will pass through (2. cor. 15. 2.) the point K.

PROP. XXVI. Prob.

To find a diameter, the centre, the axes, and the foci of an ellipsis given in position.

Draw two straight lines parallel to each other, and terminated both ways by the ellipsis; the straight line which bisects them is (4. cor. 13. 2.) a diameter; and the point bisecting that diameter is (3. 2.) the centre.

Elem.); then to the three straight lines, X, KL, and AB, find a fourth proportional (12. 6. Elem.) which will be the straight line DE wanted. For the square of X, that is, the rectangle ALB (17. 6. Elem.), is to the square of KL, (22. 6. Elem.) as the square of AB to the square of DE. Therefore (6. 2.) DE is the other axis.

Fig. 20.

In order to find the axes, find the centre C, and in the ellipsis take any point A, and join CA; and from the centre C, and with the distance AC, describe a circle AF: if this circle falls wholly without the ellipsis, CA is the greatest of the semidiameters; and therefore (9. 2.) the half of the greater axis. Next, take any point D, and let a circle be described from the centre C, with the distance CD; if this circle falls wholly within the ellipsis, CD is the least of the semidiameters; and therefore (9. 2.) the half of the less axis. Or let any other point G be taken; if the circle described from the centre C, with the distance CG, falls neither wholly without nor wholly within the ellipsis, the straight line CG is the half neither of the greater nor of the less axis; the circle, consequently, must meet the ellipsis again: let it meet it in H; and having joined GH, bisect it in K; then CK will be one of the axes, and a straight line drawn through the centre perpendicular to CK will be the other. For since GH is bisected by the diameter CK, it is parallel to the tangent AL drawn through the vertex of CK; and CKG is a right angle: therefore

CAL is also a right angle; and therefore CKA is (4. cor. 10. 2.) one of the axes. The foci are found as in prop. 24.

PROP. XXVII. PROB.

Two conjugate diameters of an ellipsis being given in position and magnitude, to find the axes, and describe the ellipsis.

Fig. 21.

Let AB, CD, be the given diameters; let them meet each other in the centre E. Suppose the problem solved, that is, let FG, HK be the axes to be found, and through A draw the straight line AL parallel to CD; AL will touch (3. cor. 14. 2.) the ellipsis in A, and will be given (28. dat.) in position: let the same AL meet the axes in the points L, M; therefore the rectangle LAM is equal to the square of CE, the semiconjugate to (21. 2.) AB: but CE is given, and, consequently, its square is given; therefore the rectangle LAM is given: let the rectangle EAN be equal to LAM; and because EA is given in position and magnitude, AN is also given in position and magnitude; and because the rectangles LAM, EAN are equal, the points L, E, M, N are (conv. 35. 3. Elem.) in the circumference of a circle: therefore, if EN is bisected in O, the centre of the circle will be in the straight line OP, which is at right angles to EN (cor. 1. 3. Elem.): but LEM being a right angle, the centre of the circle is likewise in (cor. 5. 4. Elem.) the straight line LM: it is therefore in the point P, where OP, LM intersect each other: therefore the centre P is given, and the point E is given: hence the circle described from the centre P, with the distance PE, is given in (6. def. dat.) position; so likewise are the points L, M, where its circumference meets the straight line AM given in position: therefore the axes EL, EM are given in position: draw AQ at right angles to the axis FG; and because AL touches the ellipsis, EQ, EF, EL are (17. 2.) proportionals; and EQ, EL are given: therefore EF is given in magnitude. It may in like manner be proved, that EK is given in magnitude; therefore the axes FG, HK are given in position and magnitude: and an ellipsis described through the point A, with the axis FG (25. 2.), will have AB, CD two of of its conjugate diameters.

The composition is as follows. Produce EA to N, so that the rectangle EAN may be equal to the square of CE: bisect EN in O, and draw OP at right angles to it, meeting the straight line AL, which is parallel to CE in the point P; and from the centre P, distance PE, describe a circle, and let AP meet its circumference in the points L, M: join EL, EM, and draw AQ at right angles to EL; and between EQ, EL find (13. 6. Elem.) a mean proportional EF, and make EG equal to EF, then, by the 25th proposition of this book, describe an ellipsis, of which FG may be one of the axes, and which may pass through the point A: of this ellipsis AB, CD are conjugate diameters. For since AQ is perpendicular to the axis FG, and EQ, EF, EL proportionals, AL touches the ellipsis (18. 2.) in the point A; and because CD is parallel to the tangent AL, it is in the same position with the conjugate diameter to AB; and the angle LEM being in a semicircle is a right angle; consequently EM is the other axis: hence the rectangle LAM is equal to the square of the semidiameter conjugate to AE (21. 2.): but the same rectangle LAM is equal (35. 3. Elem.) to the rectangle EAN, that is, by the construction, to the square of CE; therefore CE is the semiconjugate to AE; the ellipsis therefore passes through C; and because ED is equal to EC, and EB equal to EA, it passes likewise through the points D, B. Hence AB, CD are conjugate diameters in the ellipsis described.

PROP. XXVIII. Prob.

The position and magnitude of a diameter of an ellipsis being given, and the position of a straight line, passing through a given point in the ellipsis, and ordinately applied to that diameter being also given; to describe the ellipsis.

Fig. 21. Let AB be the given diameter, to which RS, a straight line given in position, is ordinately applied, from a given point R of the ellipsis to be described.

Bisect AB in the point E, and draw through E a straight line parallel to RS, and in that parallel take equal straight lines EC, ED, so that the rectangle ASB may be to the square of RS, as the square of AE to the square of EC or ED. If a mean proportional be found (13. 6. Elem.) between AS and SB; then (22. 6. Elem.) the mean proportional is to RS, as AE to EC, which is therefore found (12. 6. Elem.), then, by the preceding proposition, describe an ellipsis of which AB, CD may be conjugate diameters: this ellipsis will pass through the (2. cor. 15. 2.) point R, and RS will be ordinately applied (4. cor. 14. 2.) to the diameter AB.

PROP. XXIX. THEOR.

If a cone cut by a plane passing through the axis be cut also by another plane, meeting both the sides, of the triangle through the axis, but neither parallel to the base of the cone, nor subcontrarily situated; if that other plane, and the plane in which the base of the cone is situated, meet in the direction of a straight line perpendicular either to the base of the triangle through the axis, or to that base produced; the line which is the common section of this other plane, and the conical surface, is an ellipsis, which has for one of its diameters the common section of the triangle through the axis, with this same plane.

Fig. 22. Let there be a cone, the vertex of which is the point A, and the base the circle BC; let it be cut by a plane through the axis, and let the section be the triangle ABC; let it be cut likewise by another plane, meeting both the sides AB, AC, of the triangle through the axis, but neither parallel to the base of the cone, nor subcontrarily situated; let the line DEF be the common section of this other plane with the conical surface; and let GH, the common section of this plane, with the base of the cone, (continued,) be perpendicular to BC: then, the line DEF is an ellipsis: and DF, the common section of the triangle through the axis, and this same plane, is one of its diameters.

In the section DEF take any point E, and through E to DF draw EK parallel to HG; and through K draw LM parallel to BC: therefore, the plane which passes through EK, LM is parallel (15. 11. Elem.) to the plane through BC, GH, that is, to the base of the cone: consequently the plane through EK, LM (23. 1.) is a circle, of which LM is a diameter: but EK is perpendicular (10. 11. Elem.) to LM, because GH is perpendicular to BG; therefore the rectangle LKM is equal (35. 3. Elem.) to the square of EK. In like manner, any other point N being taken in the section DEF; if NO be drawn parallel to EK, or GH, and through O, PQ be drawn parallel to BC; it may be shown, that the rectangle POQ is equal to the square of NO: consequently, the square of EK is to the square of NO, as the rectangle LKM to the rectangle POQ: but (by simil. trian.) LK is to PO, as DK is to DO; and KM is to OQ, as KF is to OF; but the ratios compounded of these ratios are the same to one another; and therefore the rectangle LKM is to the rectangle POQ, as the rectangle DKF is to the rectangle DOF (23. 6. Elem.): and therefore, ex æquali, the square of EK is to the square of NO, as the rectangle DKF to the rectangle DOF. Describe, therefore, an ellipsis (28. 2.) of which DF may be a diameter, and in which EK may be ordinately applied to DF: and because the point E, by construction, is in this ellipsis, the point N is likewise in it (3. cor. 15. 2.). And the same thing may be demonstrated with regard to all the points of the section DEF.

PLATE VII.

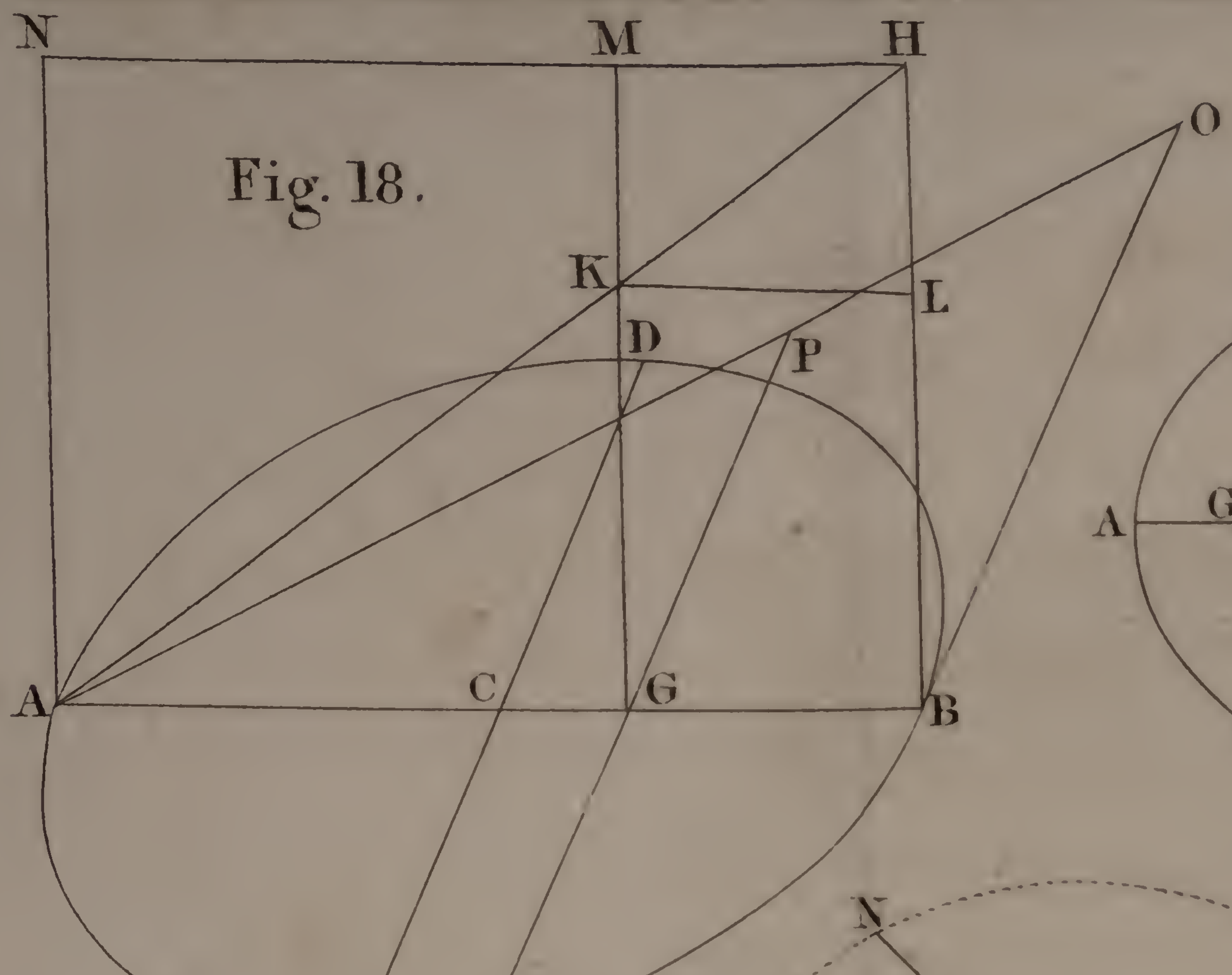
Fig. 18.

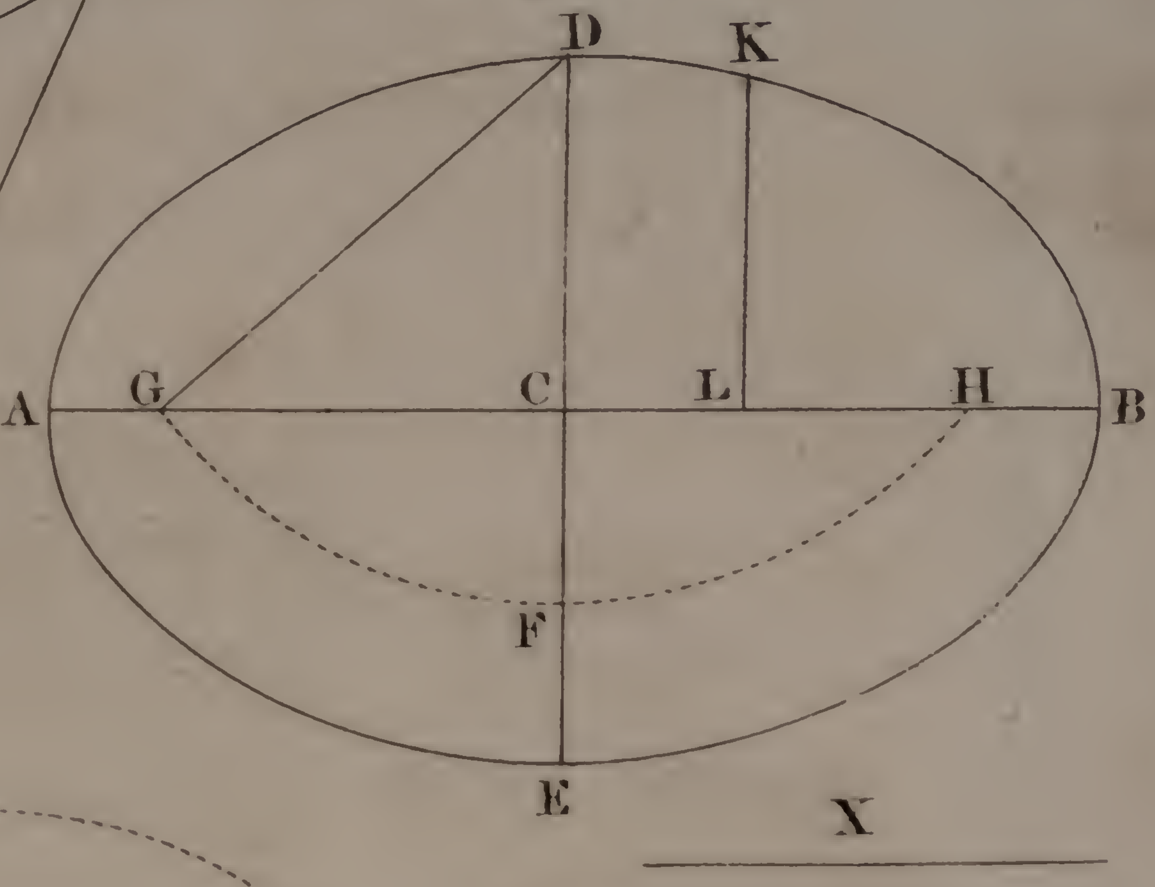
Fig. 19.

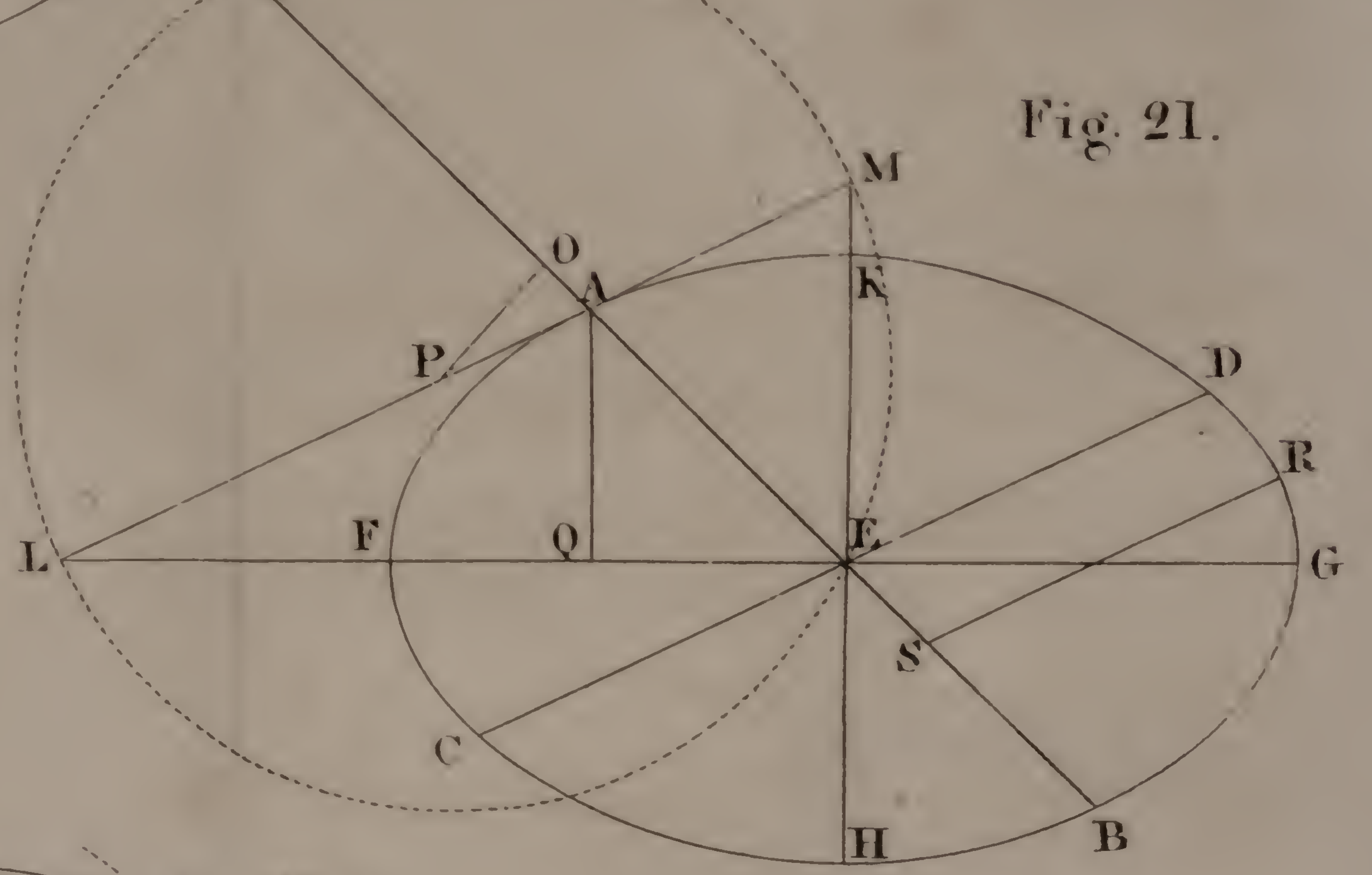
Fig. 21.

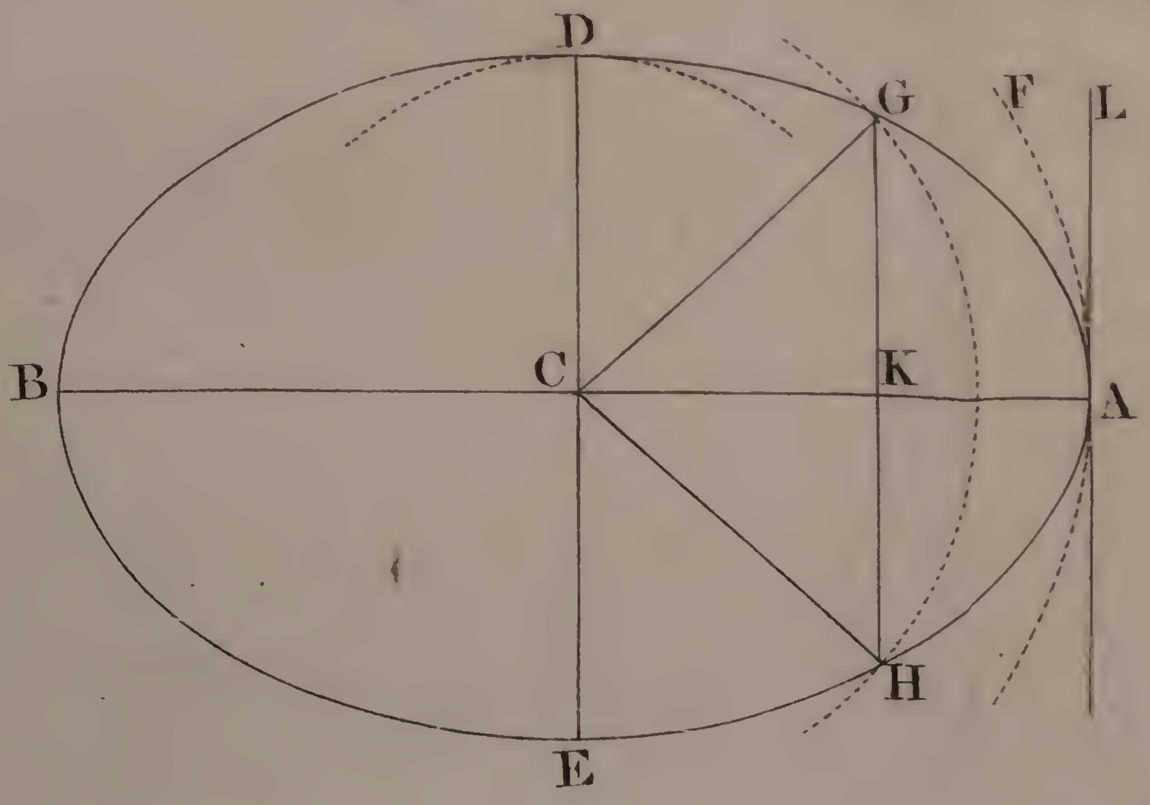
Fig. 20.

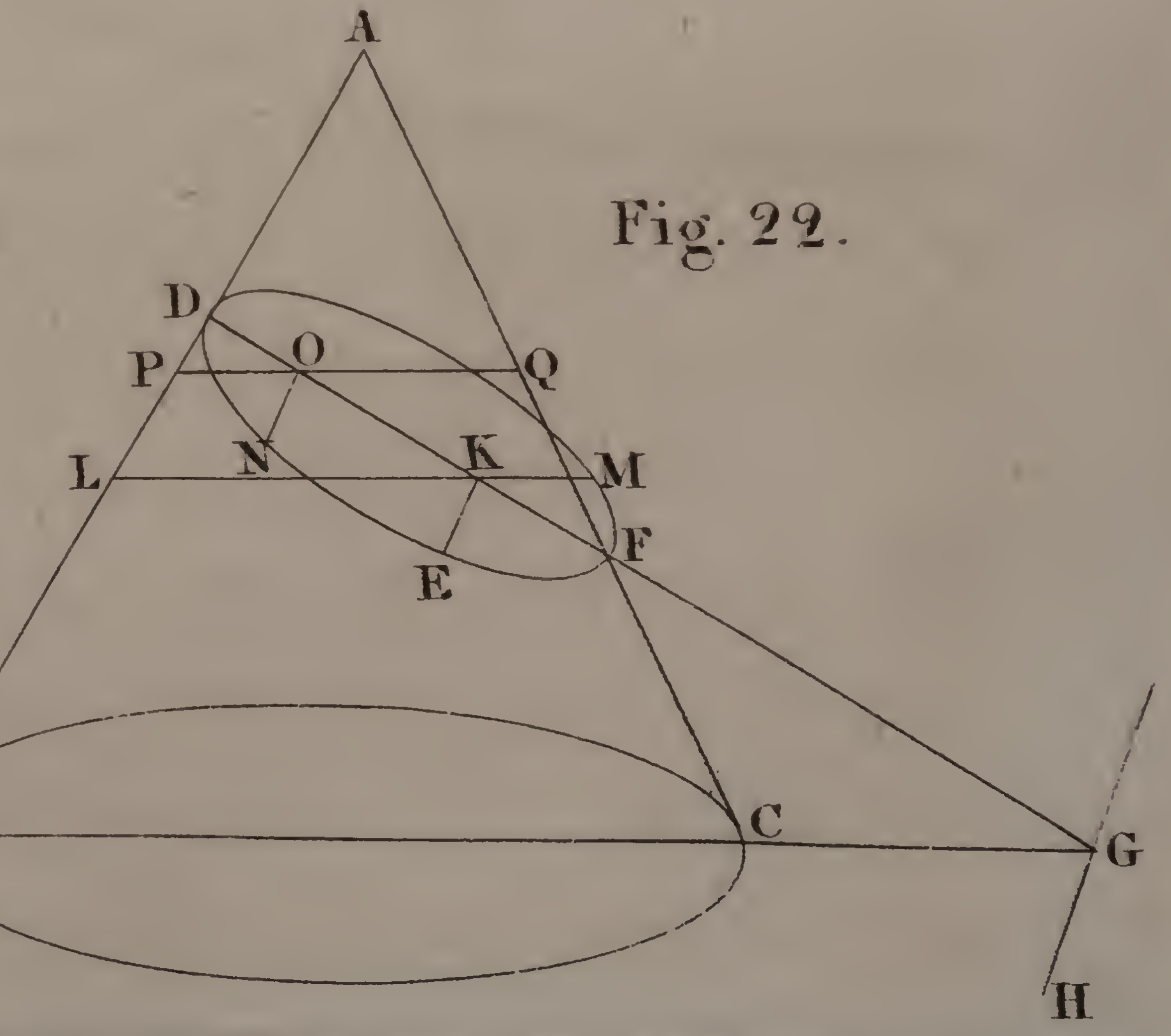
Fig. 22.

ELEMENTS

OF THE

CONIC SECTIONS.

Book Third.

OF THE HYPERBOLA.

DEFINITIONS.

BOOK III. Fig. 1.

I. IF in a point taken upon a plane, the extremity E of a ruler EH is so fixed that the ruler is left free to revolve about the point E as a centre; and if one end of a string shorter than the ruler is fixed in the extremity H, and the other end of it in the point F, which is in the same plane with the point E; but the distance between the points E, F greater than the excess of the length of the ruler above that of the string; and if by means of a pin G, the string is applied to the side EH of the ruler: then with the string so applied, and kept uniformly tense, if the ruler be moved about the centre, the point of the pin will describe upon the plane a line, called the *hyperbola.*

But if the above order be reversed, and the end E of the ruler be fixed in the point F, and the end F of the string in the point E, and then a similar operation be repeated, another line, opposite to the former, will be described, which is also called the *hyperbola;* and both together are called *opposite hyperbolas.* These lines may be extended beyond any given distance from the points E, F, if a string be taken, the length of which exceeds that distance.

II. The points E, F are called the *foci.*

III. And the point C, which bisects the straight line between the foci, is called the *centre of the hyperbola*, or of the *opposite hyperbolas.*

IV. Any straight line passing through the centre and meeting the hyperbolas, is called a *transverse diameter;* and the points where a transverse diameter meets the hyperbolas, are called its *vertices*. Also any straight line which passes through the centre, and bisects a straight line terminated by the opposite hyperbolas, but not passing through the centre, is called a *right diameter*.

V. That diameter which passes through the foci, is called the *transverse axis*.

VI. If from either extremity A of the transverse axis, a straight line AD be placed equal to the distance between the centre C and either focus F, and from A as a centre, with the distance AD, a circle be described, meeting a straight line drawn through the centre C, at right angles to the transverse axis, in the points B, *b*; the straight line B*b* is called the *second axis*. *

VII. Two diameters, each of which bisects all straight lines parallel to the other, and terminated both ways by the hyperbola, or opposite hyperbolas, are named *conjugate diameters*.

VIII. When a straight line not drawn through the centre, yet terminated both ways by the hyperbola, or opposite hyperbolas, is bisected by a diameter, it is said to be *ordinately applied* to that diameter: or it is called simply, an *ordinate* to the diameter. Also a diameter parallel to a straight line ordinately applied to another diameter, it is said to be *ordinately applied* to this other diameter.

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* Hence the second axis is bisected in the centre C (3. 3. Elem.).

IX. A straight line which meets the hyperbola in only one point, and which, being produced both ways, falls without the opposite hyperbolas, is said to *touch* the hyperbola in that point.

PROP. I. THEOR.

If from a point in a hyperbola two straight lines be drawn to the foci, the excess of the one above the other is equal to the transverse axis.

Fig. 1. Let G be a point in a hyperbola, the excess of GE above GF is equal to the transverse axis A*a*.

Let EGH represent the ruler, and FGH the string, the pin by which the hyperbola is described being supposed to remain at G; from EH, FGH take away the common part GH; and the excess of GE above GF will be equal to the excess of the length of the ruler above that of the string; and this conclusion will hold wherever the point G shall be situated in the hyperbola. And since the points A, *a*, the vertices of the transverse axis, are in the opposite hyperbolas, the excess of AE above AF; and also the excess of *a*F above *a*E, are each of them equal to the excess of the length of the ruler above that of the string, that is, to the excess of EG above FG; and therefore these two excesses are equal to each other: but let AF be added to each of the two straight lines AE, AF; and the excess of AE above AF will be equal to the excess of FE above twice AF. In like manner, the excess of *a*F above *a*E will be equal to the excess of the same FE above twice *a*E; therefore, FE exceeds twice AF by the same excess by which it exceeds twice *a*E: twice AF is, therefore, equal to twice *a*E; and therefore AF is equal to *a*E: consequently the excess of AEa bove AF is equal to the excess of AE above *a*E, that is, to the transverse axis *a*A ; and therefore the excess of EG above GF is likewise equal to the same transverse axis *a*A.

Cor. And since AF is equal to *a*E and CF to CE, therefore CA is equal to C*a*; or, the transverse axis is bisected in the centre.

PROP. II. Theor.

If from a point two straight lines are drawn to the foci of opposite hyperbolas; if the excess of the one straight line above the other be equal to the transverse axis, that point is in one of the opposite hyperbolas.

Let G be the point from whence GE, GF are drawn to the foci of opposite hyperbolas; if the excess of the one straight line above the other be equal to the transverse axis *a*A, the point G is in one of the opposite hyperbolas.

Fig. 2. Of the two straight lines let GF be the less; and from the centre F, distance FG.

describe a circle meeting FE in H; take GK equal to GF, and KE, by hypothesis, will be equal to the transverse axis Aa; and because FG, GE are together greater than FE; therefore FG, GK are together greater than FA and aE together; consequently (FG, or) FH, the half of FG, GK, is greater than FA, the half of FA, aE; and therefore the hyperbola, towards the point A, falls within the circle: and since (def. 1. 3.) it may be extended beyond any given distance from the focus F, it necessarily meets the circle. Now the hyperbola meets the circle in the point G; for if not, let it cut it in another point D, on the same side of the axis with the point G; and join DE, DF: then because the point D is in the hyperbola, (by the first Prop. 3.) the excess of DE above DF is equal to the transverse axis Aa: but by hypothesis the excess of GE above GF is equal to the same Aa; and FG is equal to FD: therefore EG is also equal to ED; which is contrary to prop. 7. b. 1. of Euclid. Therefore the point G is in the hyperbola.

PROP. III. THEOR.

If two straight lines be drawn from a point without a hyperbola to the foci, the excess of the one above the other is less than the transverse axis; but if two straight lines be drawn from a point within a hyperbola to the foci, the excess of the one above the other is greater than the transverse axis. On the contrary, any point is without, or within a hyperbola according as the excess of two straight lines drawn from that point to the foci, is less, or greater than the transverse axis.

Fig. 2. From the point L without a hyperbola, let the two straight lines LE, LF be drawn to the foci; the excess of the one above the other is less than the transverse axis A*a*. For since L is without and F within the hyperbola, the straight line LF necessarily meets

the hyperbola; let LF meet it in G, and join EG; then EL is less than EG and GL; therefore the excess of EL above LF is less than the excess of EG and GL together above the same LF, that is, than the excess of EG above GF, that is, less than the transverse axis A*a*.

Next, from the point M within the hyperbola, draw ME, MF to the foci; then ME will necessarily meet the hyperbola AG, because the point M is within and the point E without it; let it meet the hyperbola in N, and join NF. Then, because MF is less than MN together with NF, the excess of ME above MF is greater than the excess of the same ME above MN together with NF, that is, than the excess of NE above NF, that is, greater than the transverse axis A*a*.

The last part of the proposition, or the converse of these now demonstrated, is evident.

COR. Hence, if through the vertex A of the transverse axis, a straight line be drawn at right angles to that axis, this straight line is wholly without the hyperbola, and consequently touches it.

For in the straight line so drawn take any point Q, and join QF, QE; and in the axis place AR equal to AF, and join QR; then, because AR is equal to AF, that is, to *a*E, therefore RE is equal to the transverse axis A*a*; also QF is equal to QR: but QE is less than QR together with RE; and therefore the excess of QE above QR or QF, is less than RE, that is, less than the transverse axis: hence the point Q, and consequently the straight line AQ, is without the hyperbola.

PROP. IV. THEOR.

The square of half the second axis, is equal to the rectangle contained by the straight lines between either focus and the vertices of the transverse axis.

Fig. 1. Let A*a* be the transverse axis, C the centre, E, F the foci, and B*b* the second axis, which, from the definition of it, is bisected in the centre; join AB; and because AB, CF are (6. def. 3.) equal, the squares of AC, CB are together equal to the square of CF, that

is, to (6. 2. Elem.) the square of AC together with the rectangle AF*a*: take away the common square of AC, and there will remain the square of CB equal to the rectangle AF*a*.

PROP. V. THEOR.

If from a point in a hyperbola a straight line be drawn at right angles to the transverse axis, and from that point a straight line be drawn to the nearer of the foci; half the transverse axis is to the distance between that focus and the centre, as the distance between the perpendicular and the centre, is to the sum of half the transverse axis and the straight line drawn from the point to that same focus.

Fig. 3. 4.

Let G be a point in the hyperbola; from G draw GD perpendicular to the transverse Axis A*a*; and from the same point draw a straight line GF to the nearest focus F; then half the transverse axis CA, is to the distance between the centre and the focus CF, as the distance between the centre and the perpendicular CD, is to the sum of half the transverse axis and the straight line drawn to the focus, that is, to CA together with GF.

Draw GE to the other focus, and in the axis *a*A produced place AH equal to GF, and from the centre G, and distance GF, describe a circle, meeting the axis *a*A again in K, and the straight line EG in the points L, M: and because EF is the double of CF, and FK the double of FD, therefore EK is the double of CD. Again, because EL or A*a* is the double of CA, and LM the double of GF or AH, therefore EM is the double of CH: but, on account of the circle, EL or A*a* is to EF, as EK to EM (cor. 36. 3. and 16. 6. Elem.); take the halves of these proportionals, and CA will be to CF, as CD to CH.

PROP. VI. Theor.

Fig. 3. 4. The same construction remaining, if from A, the vertex of the transverse axis nearest to G, and in this same axis produced, a part AH be taken equal to the distance between the point G and the focus F ; the square of the perpendicular GD is equal to the excess of the rectangle EHF, contained by the segments between the point H and the foci, above the rectangle AD*a*, contained by the segments between the perpendicular and the vertices of the transverse axis.

For since the straight line CH is cut into two parts in the point A, the squares of CA, CH are together equal to twice the rectangle ACH, together with the square of AH (7. 2. Elem.) that is, because CA, CF, CD, CH are proportionals, (preced. prop.) equal to twice the rectangle FCD, together with the square of AH or GF, that is, equal to twice the rectangle FCD, together with the squares of FD, DG, that is, equal to the sum of the squares of FC, CD and DG (7. 2. Elem.): therefore the two squares of CA, CH are equal to the three squares of CF, CD, DG: but the sum of the two first is equal (6. 2. Elem.) to the squares of CA, CF, together with the rectangle EHF; and the sum of the three last is equal (6. 2. Elem.) to the squares of CA, CF, DG, and the rectangle AD*a*: from these equals take away the common squares of CA, CF, and there will remain the rectangle EHF, equal to the square of DG, together with the rectangle AD*a*.

PROP. VII. THEOR.

If from a point in a hyperbola a straight line be drawn perpendicular to the transverse axis; the square of the transverse axis is to the square of the second axis, as the rectangle contained by the segments between the perpendicular and the vertices of the transverse axis, is to the square of the perpendicular.

Fig. 3. 1. Let G be a point in the hyperbola; from G draw GD perpendicular to the transverse axis A*a*; then the square of A*a* is to the square of B*b*, as the rectangle AD*a*, contained by the segments between the vertices of the transverse and the point D, is to the square of GD.

Having drawn the straight lines GE, GF to the foci, place AH from the nearest vertex to the focus F, of the transverse axis, equal to GF the lesser of them: then, because CH, CD, CF, CA are proportionals; their squares are also proportionals; but the square of CH, is equal to the square of CF together with the rectangle EHF, and the square of CD is equal to the square of CA together with the rectangle AD*a* (6. 2. Elem.); therefore, as the whole square of CH is to the whole square of CD, so is the square of CF taken from the first, to the square of CA taken from the second: therefore the remaining rectangle EHF is to the remaining rectangle AD*a*, as the square of CF to the square of CA (cor. 19. 5. Elem.); and, by division, the excess of the rectangle EHF above the rectangle AD*a*, is to AD*a*, as the (6. 2. Elem.) rectangle AF*a* to the square of CA: but (by the preceding prop. and 4. 3.) the square of GD is to the rectangle AD*a*, as the square of CB is to that of CA; and, inversely, the square of CA is to the square of CB, as the rectangle AD*a* is to the square of GD.

Cor. The squares of straight lines drawn perpendicular to the transverse axis from points in a hyperbola, or in opposite hyperbolas, are to one another, as the rectangles contained by the segments intercepted between those straight lines and the vertices of the transverse axis; as was shown in the ellipsis (1. cor. 6. 2.).

PROP. VIII. THEOR.

If from a point in a hyperbola a straight line be drawn perpendicular to the second axis; the square of the second axis is to the square of the tranverse, as the sum of the squares of half the second axis, and of its segment between the perpendicular and the centre, is to the square of the perpendicular.

Fig. 4.

From a point G of a hyperbola, draw GN perpendicular to the second axis B*b*; the square of B*b* is to the square of A*a*, as the sum of the squares CB, CN, to the square of GN.

Because, by the preceding, the square of CA is to the square of CB, as the rectangle AD*a* is to the square of GD: therefore, inversely, and by proposition 12. B. 5. Elem. the square of CB is to the square of CA, as the sum of the squares of CB, GD is to the square of CA, together with the rectangle AD*a*, that is, as the sum of the squares of CB, CN is to the square of CD or GN.

Cor. Hence, if from two points of a hyperbola, or of opposite hyperbolas, perpendiculars be drawn to the second axis, the square of the one perpendicular is to the square of the other, as the sum of the squares of half the second axis, and of the distance between the former perpendicular and the centre, is to the sum of the squares of half the second axis, and of the distance between the latter and the centre.

PROP. IX. THEOR.

A straight line terminated both ways by a hyperbola, or opposite hyperbolas, and parallel to either axis, is bisected by the other axis; or, what is the same thing, the axes, are conjugate diameters.

Fig. 5. First, let the straight line DE be parallel to the second axis B*b*, and meet the transverse in F; and thus the square of DF is to the square of EF as the rectangle AF*a* is to the (cor. 7. 3.) rectangle AF*a*: therefore DF, FE are equal.

Next, let DG be parallel to the transverse axis A*a*, and meet the second axis B*b* in K; and thus the square of DK is to the square of KG, as the sum of the squares of CB, CK is to the sum of the same squares (cor. preced.) of CB, CK: therefore DK, GK are equal.

PROP. X. THEOR.

A straight line terminated both ways by a hyperbola, or opposite hyperbolas, and

PLATE VIII.

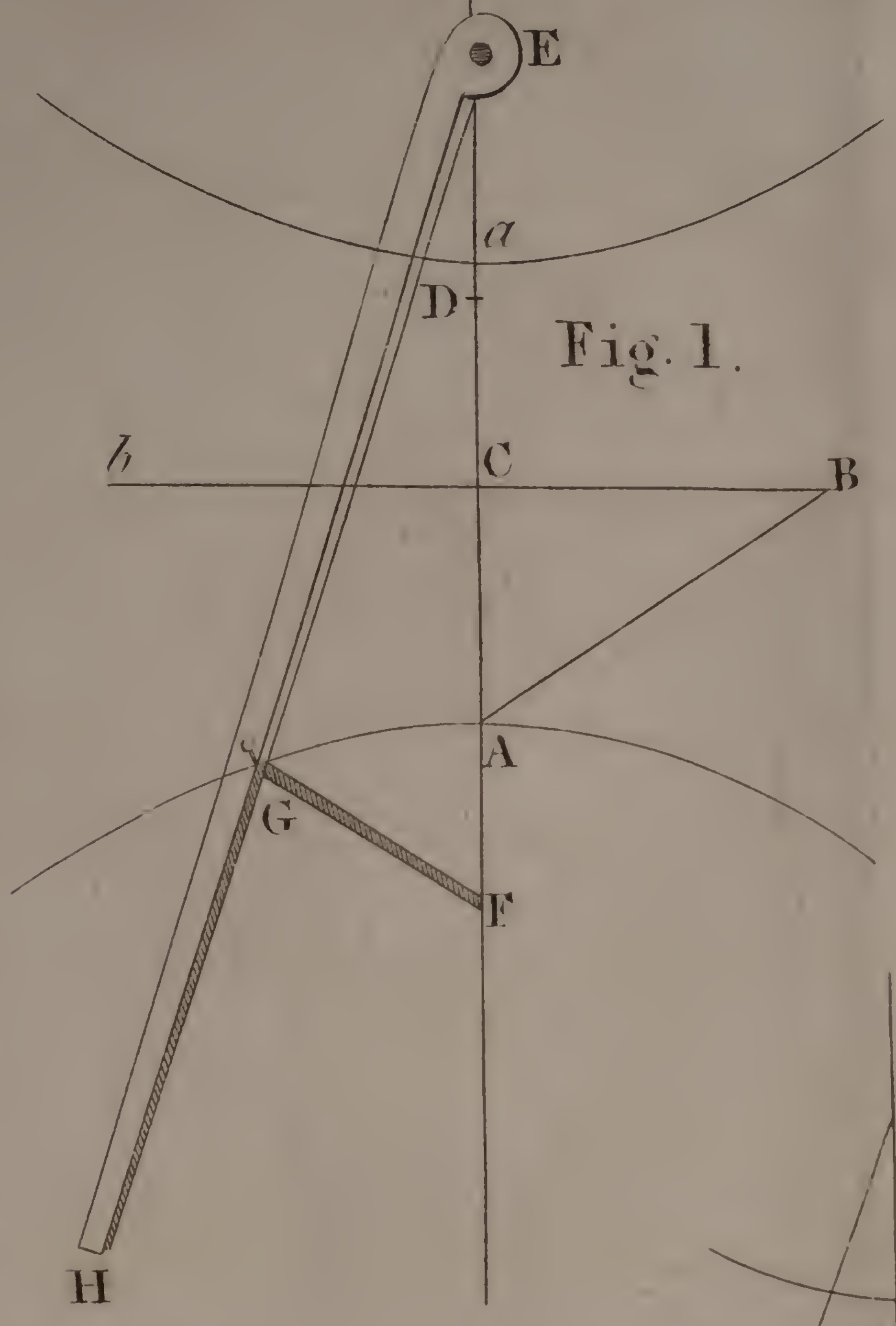


Fig. 1.

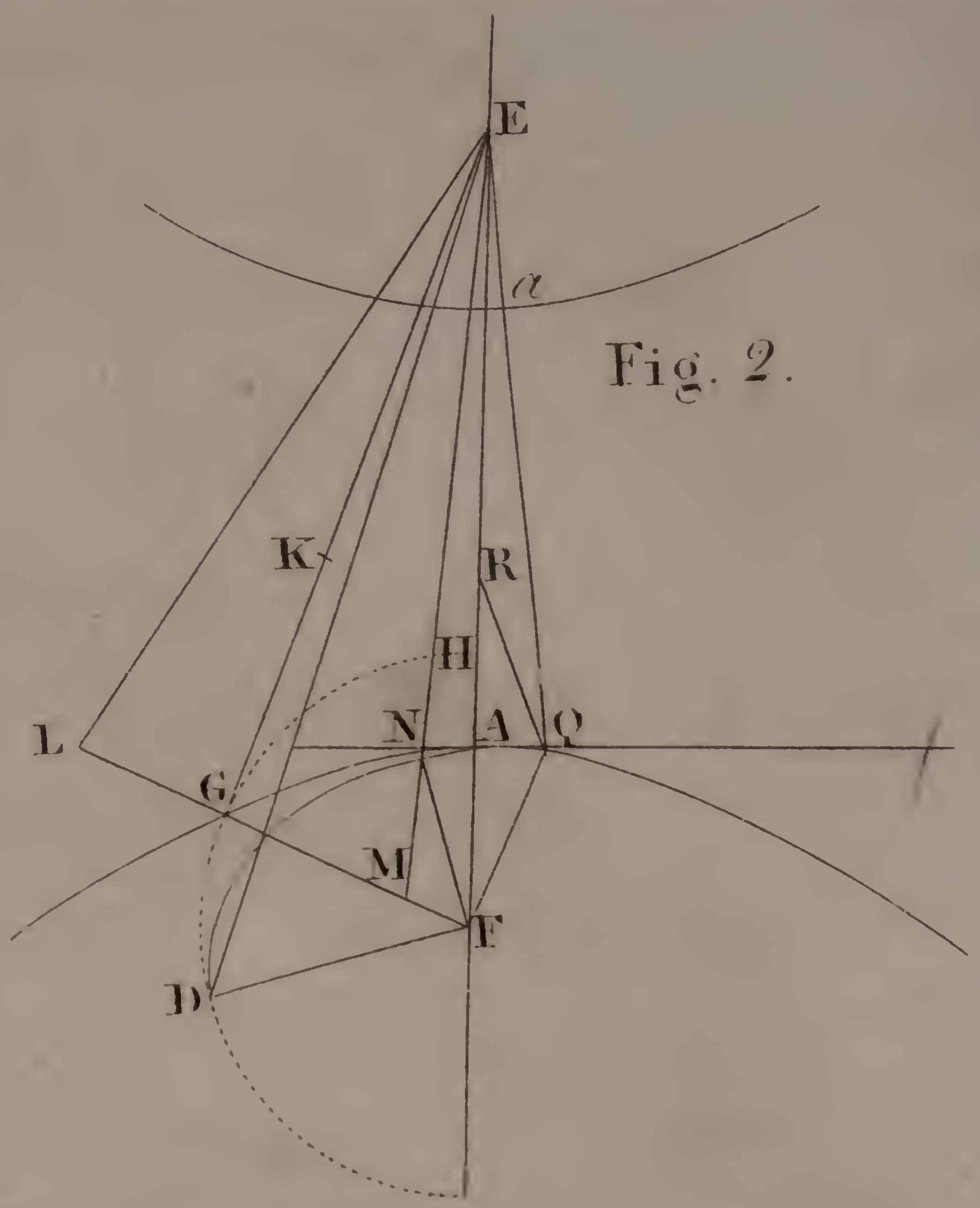


Fig. 2.

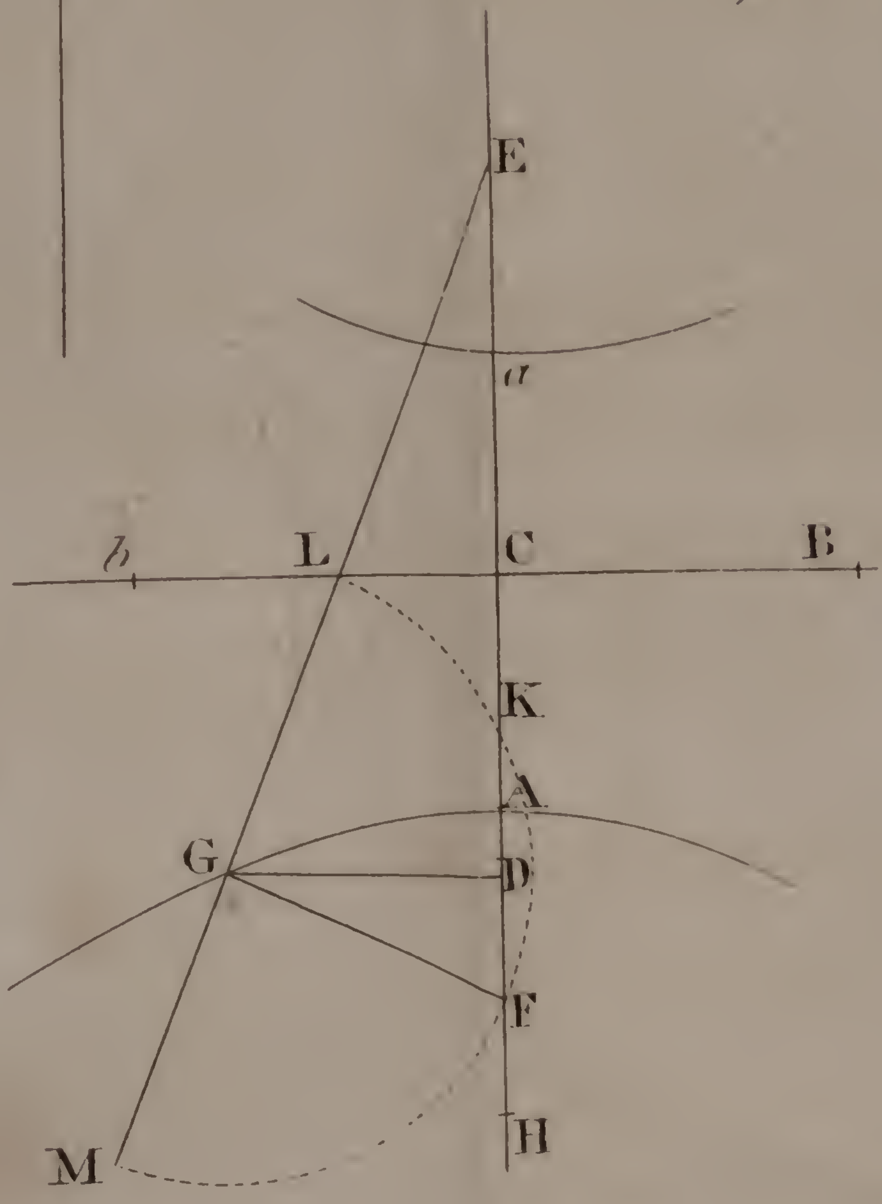


Fig. 3.

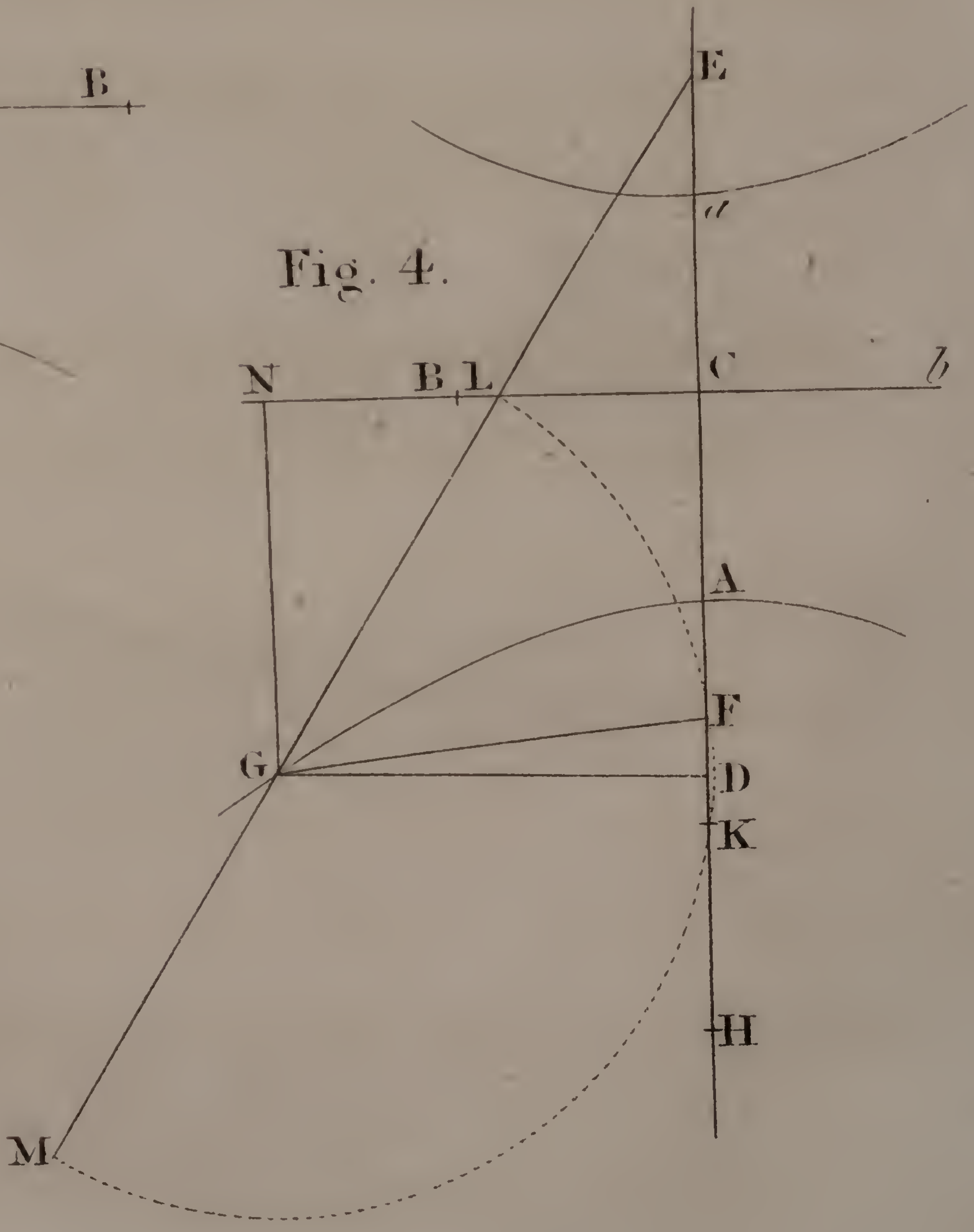
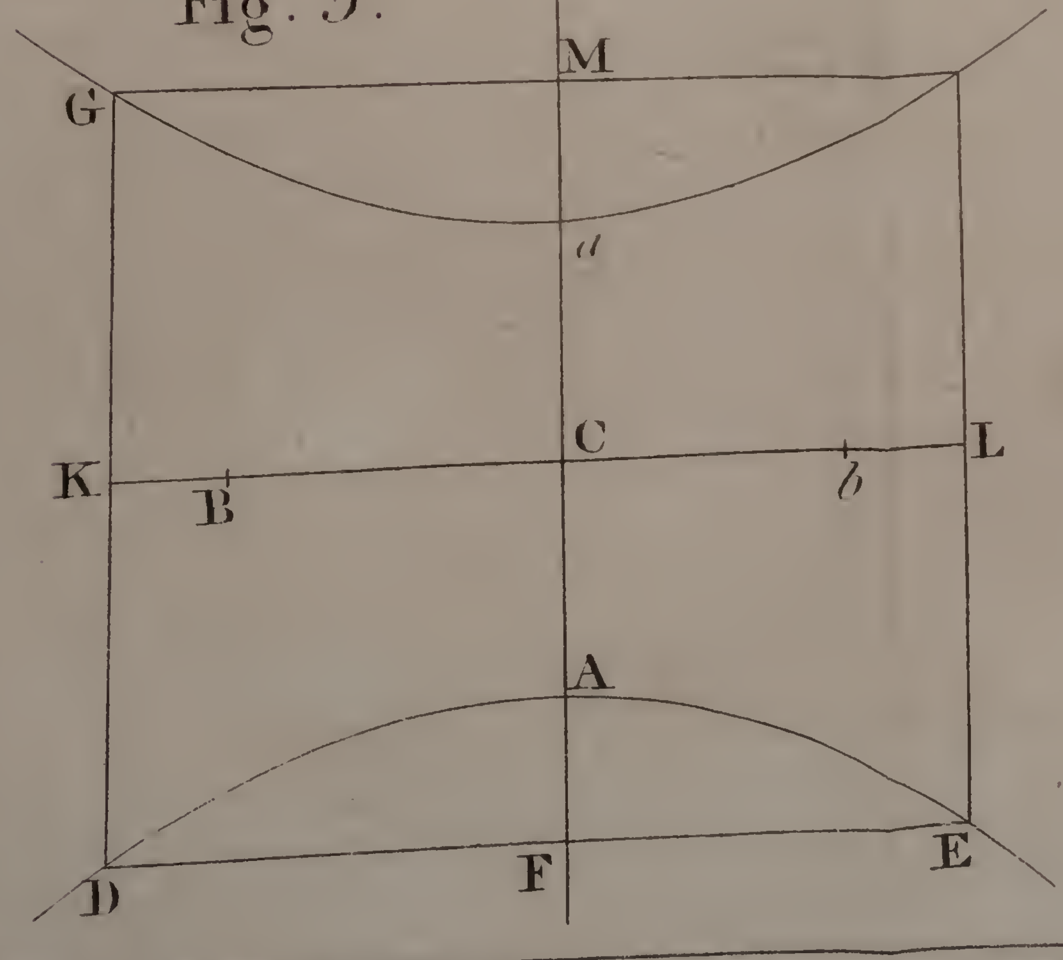


Fig. 4.

Fig. 5.



bisected by either axis is parallel to the other axis.

BOOK III.

First, let DE be bisected by the transverse axis in F; and draw DK, EL parallel to the same axis, and meeting the second axis in the points K, L; then, because DF, FE are equal, KC, CL are also equal: but the square of DK is to the square of EL, as the squares of CB, CK together, to the squares of CB, CL together; therefore DK, EL are equal, and they are parallel: consequently DE, KL are also parallel (33. 1. Elem.).

Fig. 5.

Next, let DG be bisected by the second axis in the point K, and draw DF, GM parallel to the same axis, and meeting the transverse axis in F, M; then, because DK, GK are equal, FC, CM are likewise equal; and, of consequence, FA, *a*M are equal: now the square of DF is to the square of GM, as the rectangle AF*a* to the rectangle AM*a*; but the rectangles AF*a*, AM*a* are equal; and therefore the straight lines DF, GM are equal, and they are parallel; consequently DG, FM are likewise parallel (33. 1. Elem.)

Cor. It is manifest from the demonstration, that the straight lines DF, GM, which are parallel to either axis B*b*, and cut off, between the centre and the points where they meet the other axis, equal segments FC, MC, are also equal. In the same manner, DK, EL are equal, which are parallel to the axis A*a*, and cut off the equal segments CK, CL.

And the contrary: if DF, GM are equal to each other, and parallel to B*b*, they cut off equal segments FC, MC. In like manner, if DK, EL be equal to each other, and parallel to A*a*, they cut off equal segments CK, CL.

PROP. XI. Theor.

Any straight line perpendicular to the transverse axis, and meeting it below the vertex, will meet the hyperbola in two points.

Fig. 6. 7. Let DC be perpendicular to the transverse axis A*a*, and meet it in C, below the vertex A; then DC meets the hyperbola in two points. Let E, F be the foci; and from C, place CG equal to CF, the distance between

C and the nearest focus; and from the other focus place EK, equal to the transverse axis A*a*. If then, the point C be below the focus F, it is evident, that EK is less than EG: but in the other case where the point C is above F; since A*a*, EK are equal, AK is equal to *a*E, that is, to AF; and, by hypothesis, FC is less than FA; twice FC is, therefore, less than twice FA, that is, FG is less than FK; and thus EK is less than EG: make then as EK to EF, so is EG to a fourth proportional EH; and since EK is less than each of the two EF, EG, and of consequence, much less than EH; therefore EK, EH are (25. 5. Elem.) together greater than EF, EG together. From these unequals take away twice EK, and KH will be greater than KF and KG together, that is, than twice KC; for CF is equal to CG: hence, if KH is bisected in L, KL will be greater than KC; and therefore the point L falls below the straight line CD; and a circle described from the centre E, with the distance EL, will necessarily meet CD in two points D, *d*. Describe, from the centre D, distance DF, another circle, which (3. 3. Elem.) will pass

Fig. 6.

Fig. 7

Fig. 6. 7.

through the point G; join DE, and let this circle meet it in the points M, N; and because EK is to EF, as EG to EH, the rectangle HEK is equal to the rectangle FEG, that is, to the rectangle MEN (cor. 36. 3. Elem.); and ED, EL are equal; and thus their squares are equal; from which take away the equal rectangles MEN, HEK, and the remaining square of DM, or DF is equal to the remaining square of KL (6. 2. Elem.); consequently DM and KL are equal, and which being taken from the equals ED, EL, the remainder EM is equal to the remainder EK, or the transverse axis A*a*; and EM is the excess of DE above DF; therefore the point D is in the hyperbola. In like manner it may be demonstrated that the point *d* is in the hyperbola.

DEFINITION X.

Fig. 8. If through one of the vertices of the transverse axis a straight line be drawn equal and parallel to the second axis, and bisected by the transverse axis; the straight lines drawn

through the centre and the extremities of the parallel are called the *asymptotes*.

Cor. 1. The asymptotes of two opposite hyperbolas are common to both.

For let CD, CE be the asymptotes of the hyperbola AF, and draw through the vertex A of the transverse axis the straight line DAE parallel to the second axis B*b*, and through the other vertex *a* the straight line *dae* parallel to DE; then because CD, CE are asymptotes, DA, AE are each of them equal and parallel to CB, half the second axis: and because DE, *de* are parallel, and CA, C*a* equal, (by simil. trian.) *ad*, *ae* are equal and parallel to AD, AE; consequently they are equal and parallel to half the second axis: therefore C*d*, C*e* the continuations of CD, CE, are also asymptotes of the opposite hyperbola *a*.

Cor. 2. The asymptotes are parallel to straight lines joining the extremities of the axes; for if AB, *b*A be joined, CE, CD are parallel to them (33. 1. Elem.).

PROP. XII. THEOR.

The asymptotes do not meet the hyperbola.

Fig. 8. Let there be a hyperbola, the transverse axis of which is A*a*, and the centre C; and through A draw a straight line perpendicular to CA, and in this perpendicular take AD, AE, equal, each of them, to half the second axis; join CD, CE; which are therefore the asymptotes: now if possible, let CD meet the hyperbola in F, and through F draw a straight line parallel to DA, and meeting the axis A*a* in G; and since the rectangle AG*a* is to the square of GF, as the square of CA is (7. 3.) to that of CB or AD, that is, as the square of CG is to that of GF, therefore the rectangle AG*a* is equal to the square of CG; which is absurd (6. 2. Elem.): the asymptote, therefore, meets not the hyperbola in F. In like manner it may be shown, that it does not meet the hyperbola in any other point.

PROP. XIII. THEOR.

If through a point of a hyperbola a straight line be drawn parallel to the second axis, and meeting the asymptotes; the rectangle contained by its segments intercepted between the asymptotes and that point, is equal to the square of half the second axis.

Let F be a point in the hyperbola; through F draw KFL parallel to the second axis, and meeting the asymptotes in the points K, L; the rectangle KFL, is equal to the square of CB. Fig. 8.

Through the vertex A of the transverse axis draw DAE meeting the asymptotes in the points D, E; and let KL meet the same axis in G: therefore AD, AE are each of them equal and parallel to half the second axis. To the second axis draw the straight line FM parallel to CA; and, by prop. 8. of this book, the square of CB or AD will be to the square of CA, as the sum of the squares of CB, CM is to the square of FM or GC; and (by simil. trian.) the square of AD is to the square of AC, as the square of KG to the square of GC; therefore the sum of the squares of CB, CM is to the square of GC, as the square of KG is to the same square of GC: consequently the sum of the squares of CB, CM is equal to (9. 5. Elem.) the square of KG: from these equals take the equal squares of CM, FG, and the remaining square of CB is (5. 2. Elem.) equal to the remaining rectangle KFL. In like manner, if KL meets the hyperbola again in H, it may be shown, that the rectangle KHL is equal to the square of CB.

Cor. Hence if in a straight line KL terminated by the asymptotes, and parallel to the second axis, there be taken a point F, so situated, that the rectangle KFL may be equal to the square of the second axis; that point is in the hyperbola.

PROP. XIV. Theor.

If a straight line meeting a hyperbola, or the opposite hyperbolas in two points, meets also the asymptotes; the rectangle contained by the segments between the asymptotes and the one point, is equal to that contained by the segments between the same asymptotes and the other point: and the straight lines intercepted between the asymptotes and the points in the hyperbola are equal.

Let AB be a straight line meeting the hyperbola, or opposite hyperbolas, in the points A, B, and the asymptotes in C, D; then the rectangles CAD, CBD are equal; and also CA, BD are equal. Fig 9.

Through the points A, B draw straight lines parallel to the second axis, and meeting the asymptotes in E, F and in G, H: and since, by the preceding proposition, the rectangles EAF, GBH are each of them equal to the square of half the second axis, they are equal to each other; therefore, as EA to GB, so is BH to AF: but the triangles being equiangular, as EA to GB, so is CA to CB; and as BH to AF, so is BD to AD: therefore as CA to CB so is BD to AD; therefore the rectangle CAD is equal to the rectangle CBD: take away, or add the common rectangle AC, BD, according as the points are in the same, or in opposite hyperbolas, and the rectangle CAB is equal to the rectangle ABD; and therefore the straight lines AC, BD are equal (1. 6. Elem.).

Cor. If from two points A, L in a hyperbola, to either asymptote KC straight lines AM, LN be drawn parallel to the other asymptote; and from any other point B in the hyperbola the straight lines AB, LB be drawn meeting the same asymptote KC in C, O; then CO, MN are equal. For let AB, LB meet the asymptote KP in the points D, P, and to the other asymptote KC draw BQ parallel to the asymptote KP: because AC, BD are equal, as also OL, BP; therefore CM, QK are equal, as also ON, QK: consequently

CM is equal to ON; and MO being common, CO, MN are likewise equal.

PROP. XV. THEOR.

If through two points in a hyperbola, or in opposite hyperbolas, two parallel straight lines be drawn which meet the asymptotes; the rectangles contained by their segments between the points and the asymptotes are equal

Fig. 10. n. 1.

Let A, B be two points in a hyperbola, or in opposite hyperbolas; through these points draw CD, EF parallel to each other, and meeting the asymptotes OC, OD in the points C, D and E, F; the rectangles CAD, EBF are equal.

Through the points A, B to the asymptotes draw the straight lines GAH, KBL parallel to the second axis; and because the rectangles GAH, KBL are each of them equal (13. 3.) to the square of half the second axis, they are equal to each other: therefore GA is to KB, as BL to AH: but the triangles GAC, KBE being equiangular, GA is to KB, as CA to EB; and the triangles LBF, HAD being equiangular, BL is to AH, as BF to AD: therefore CA is to EB, as BF to AD; and consequently the rectangles CAD, EBF are equal.

Fig. 10. n. 2. COR. 1. And if through the centre a straight line AOM be drawn meeting both the hyperbolas, and parallel to the straight line BEF; the square of either segment AO, intercepted between the centre and either hyperbola, is equal to the rectangle EBF. The demonstration is the same as in the proposition.

COR. 2. Hence any straight line drawn through the centre, and terminated by opposite hyperbolas, is bisected in the centre.

Fig. 10. n. 1. COR. 3. If CD, EF meet a hyperbola, or its opposite hyperbola again in the points M, N, the rectangle ACM or ADM, is equal to the rectangle BEN or BFN: for AC, MD are equal, as also BF, NE.

Fig. 10. n. 2. COR. 4. And since it has been proved, that the square of the semidiameter AO or OM is equal to the rectangle EBF, that is, to BEN; therefore BE is to AO, as AO to EN; and

PLATE IX.

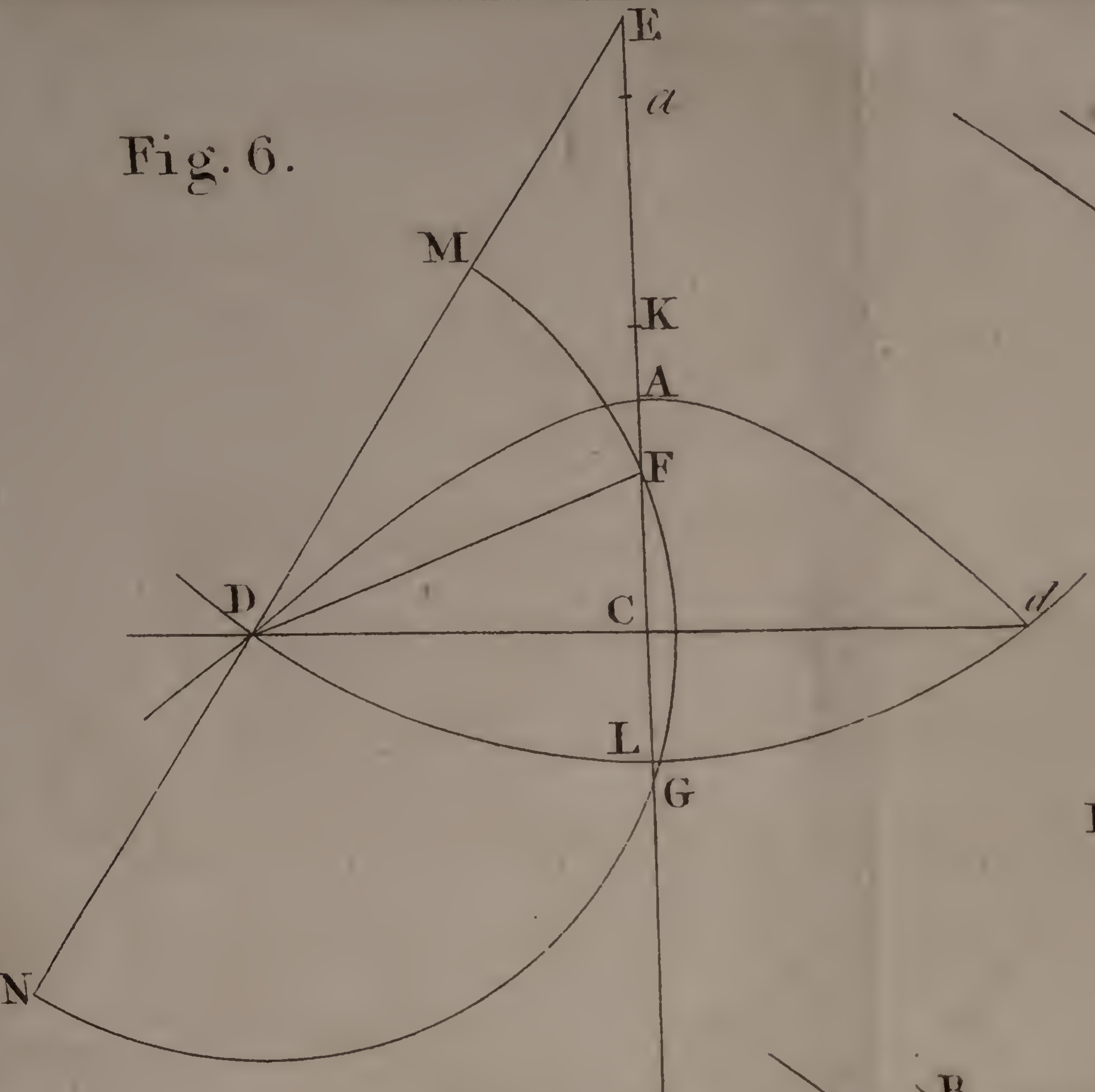


Fig. 6.

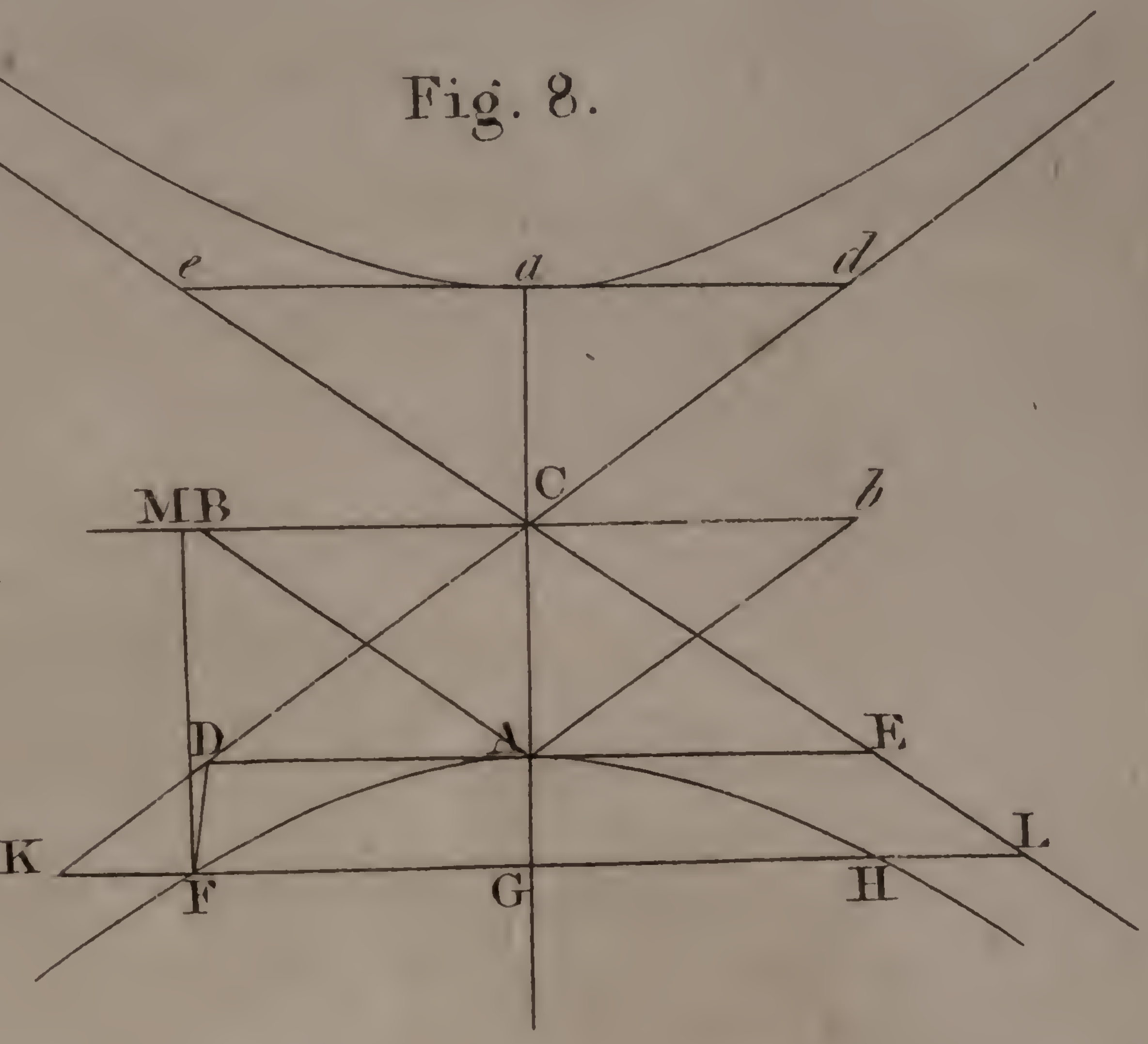
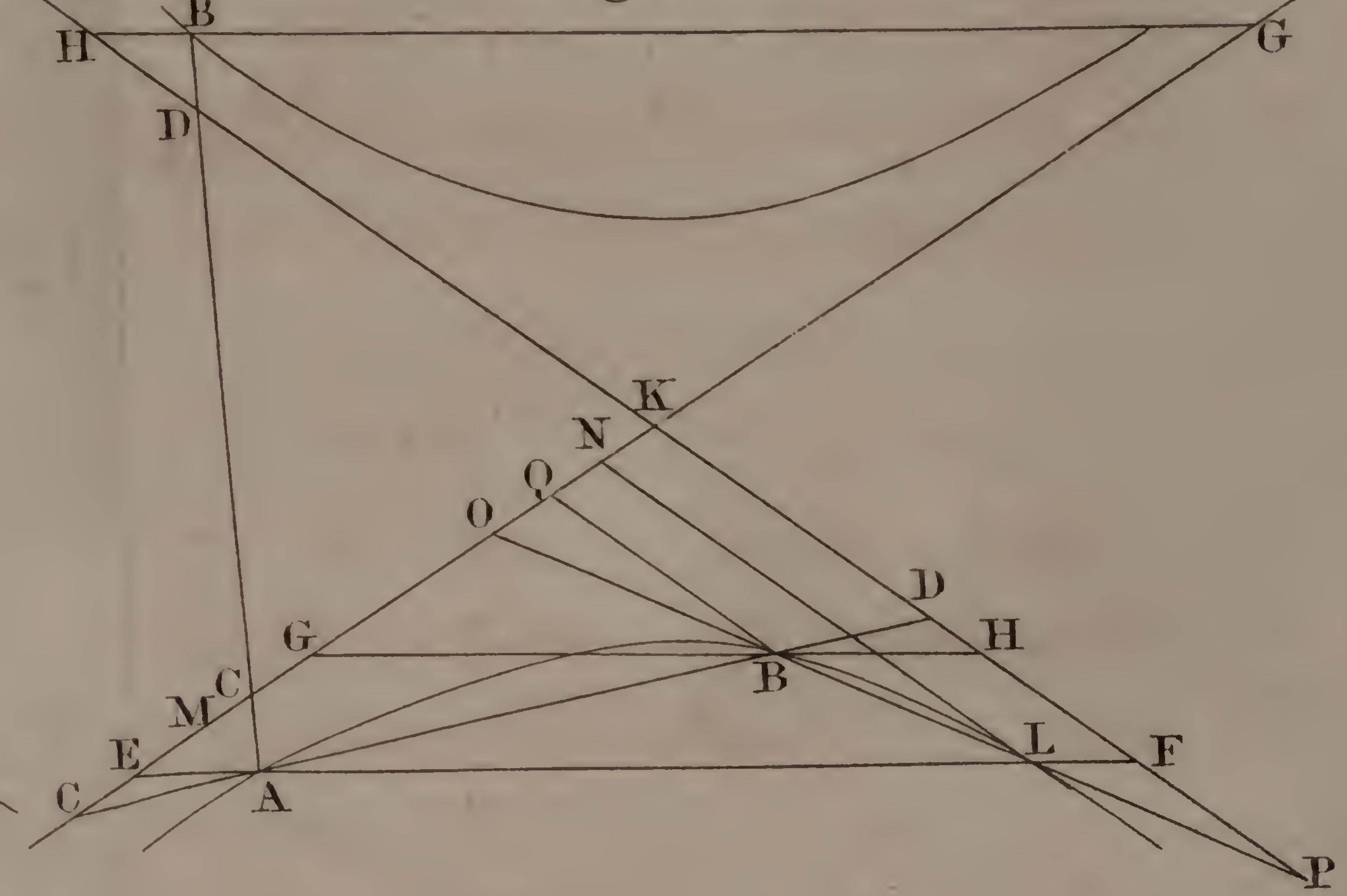


Fig. 8.

Fig. 9.



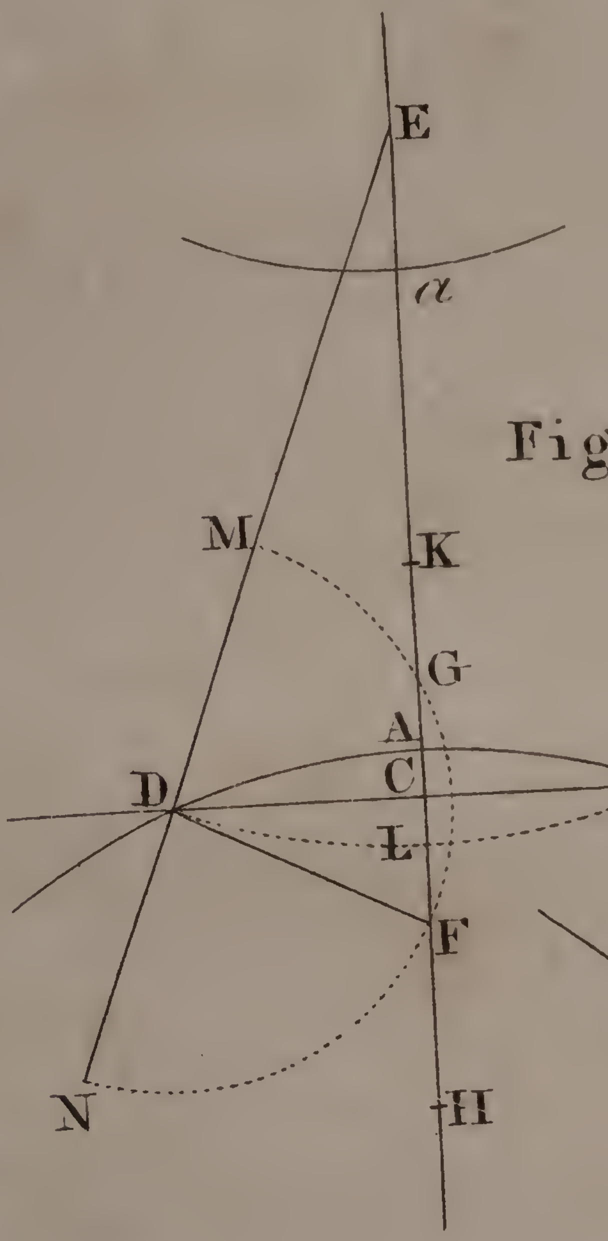


Fig. 7.

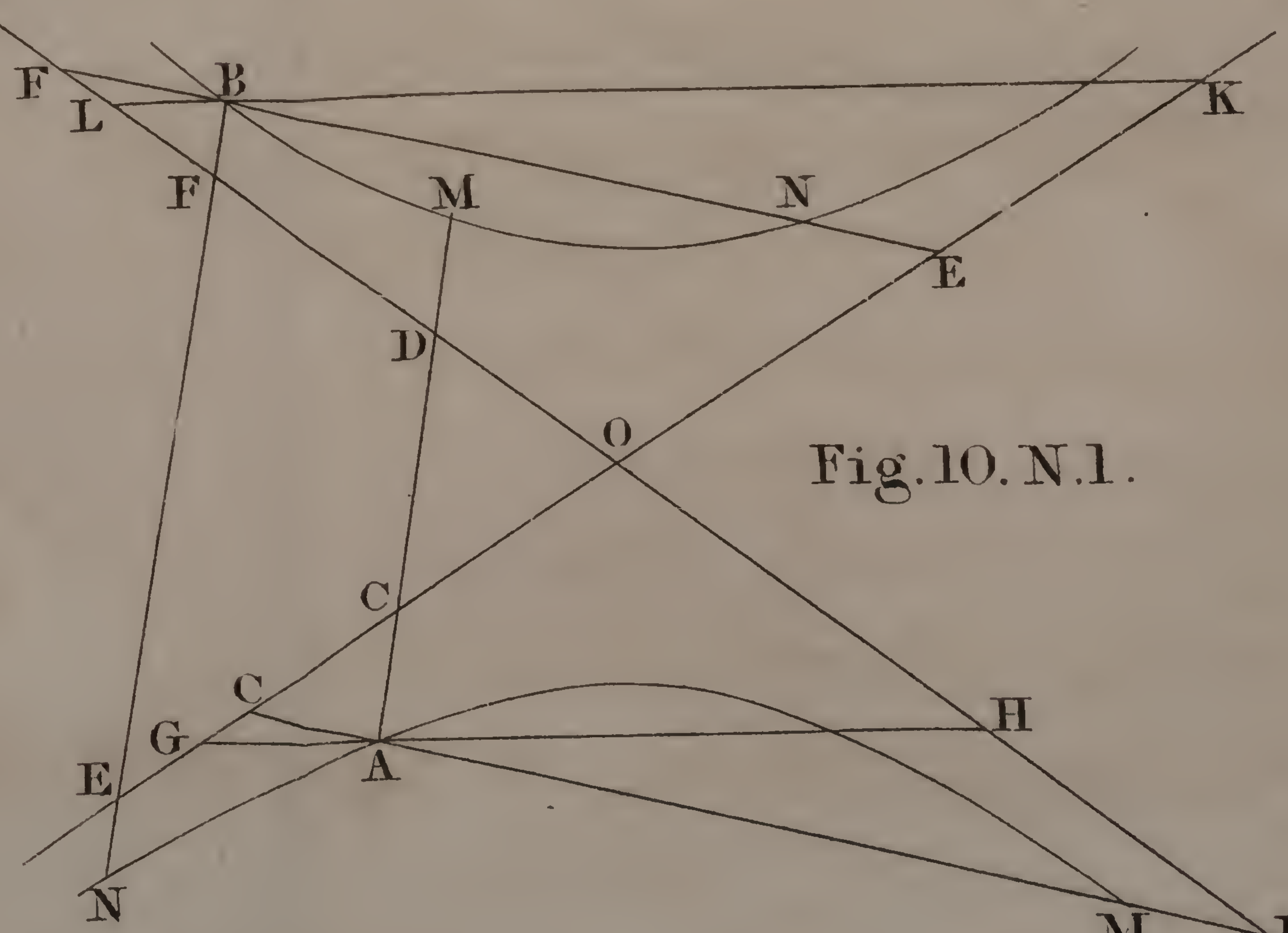


Fig. 10. N. 1.

consequently BN is greater than (25. 5. Elem.) twice AO, that is, than AM; that is, any transverse diameter is less than any other straight line parallel to it, and terminated in opposite hyperbolas.

Cor. 5. If in a straight line BN terminated by the hyperbolas, there be taken the points E, F such, that each of the rectangles BEN, BFN be equal to the square of the semidiameter AO, which is parallel to BN; the points E, F are in the asymptotes.

PROP. XVI. Theor.

If from a point in a hyperbola to the asymptotes any two straight lines be drawn, to which other two straight lines drawn to the asymptotes from any other point in the same, or opposite hyperbola, are parallel; the rectangle contained by the former straight lines is equal to that contained by the latter.

Let A, B be points in a hyperbola, or in opposite hyperbolas; through A draw the Fig. 11.

straight lines AC, AD to the asymptotes, and through B draw BE, BF parallel to AC, AD: the rectangle CAD is equal to the rectangle EBF.

Draw through the points A, B to the asymptotes the straight lines GAH, KBL parallel to the second axis, and the proposition may be demonstrated in the same words with the preceding.

Cor. 1. Hence, if from two points in a hyperbola, or opposite hyperbolas, to one or both of the asymptotes, two straight lines be drawn parallel to the other asymptote, or to both of them; the rectangles contained by each parallel and the abscissa * between it and the centre are equal.

Let A, B be the points; through them draw AC and BE, or BF, parallel to the asymptotes; the rectangle contained by the parallel AC, and the abscissa CO between AC

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* In general, the parts cut off in an indefinite straight line, and estimated from a given point, by parallels drawn from any curve line, and forming with it a given angle, are called *Abscissæ*, or abscissas; and the parallels are called *Ordinates* to that curve.

and the centre, is equal to the rectangle BEO or BFO. For complete the parallelograms ACOD, BEOF and the rectangles CAD, EBF, that is, the rectangles ACO, BEO, will be equal.

Cor. 2. And since the rectangles CAD, EBF are equal, AC is to BE, as BF to AD; and the parallelograms ACOD, BEOF being equiangular, they are therefore equal (14. 6. Elem.).

PROP. XVII. Theor.

Any straight line drawn through the centre, and within the angle contained by the asymptotes, meets the hyperbola.

Let AB, AC be the asymptotes, and AD the half of the transverse axis, and let AE be any straight line drawn from the centre, and passing within the angle BAC; this straight line AE meets the hyperbola. For if AE meets not the hyperbola, through D draw BDC parallel to the second axis, and meeting the asymptotes in B, C; draw also DF parallel to AB, and let DF meet AE in F; and

Fig. 12.

having taken BG equal to DF, join GF, which will be equal and parallel to BD, and will therefore meet the transverse axis at right angles, and so will cut the hyperbola (11. 3.): let it cut it in H on the same side of AD with the point F; and let it meet the other asymptote in K: since, therefore, the point F is without the hyperbola, GH is greater than GF, that is, than BD; and HK is greater than DC: therefore the rectangle GHK is greater than the rectangle BDC, that is, than the square of BD: but, by the 13th of this book, the rectangle GHK is equal to the square of BD; which is absurd. Therefore AE necessarily meets the hyperbola.

Fig. 10. n. 2. Cor. If from the centre a straight line OA be drawn within the angle contained by the asymptotes; and the square of that straight line be equal to the rectangle EBF, contained by the segments of any straight line parallel to OA, which are intercepted between the point B, where that parallel meets the hyperbola, and the points E, F where it meets the asymptotes; the point A is in one of the hyperbolas. For, according to the proposition, the straight line OA necessarily meets the hyperbola, if, therefore, it meets not the hyperbola in A, it must meet it in some other point P; and then the square of OP will be equal to the (1. cor. 15. 3.) rectangle EBF, that is, to the square of OA; which is absurd. Therefore the point A is in the hyperbola.

PROP. XVIII. *(Prop. 13. B. 2. Apoll.)*

If within the angle contained by the asymptotes, any straight line be drawn parallel to either of the asymptotes; it meets the hyperbola in one point only, and passes within the hyperbola.

Fig. 12.

Let there be a hyperbola, the asymptotes of which are AL, AM; take any point N, and through N draw NO parallel to AL; the straight line NO will meet the hyperbola. For, if possible, let NO not meet the hyperbola; in the hyperbola take any point P, through which draw PQ, PM parallel to AM, AL; and make the rectangle ANO equal to MPQ; and having joined AO, produce it, AO will meet the hyperbola (17. 3.): let it meet it in the point R, and through R draw RS, RT parallel to AM, AL; therefore the rectangle MPQ is equal to the (16. 3.) rectangle TRS: but the rectangle ANO is made equal to the same MPQ; consequently the rectangle TRS, that is, the rectangle ATR, is equal to ANO; which is impossible; because RT is greater than NO, and AT greater than AN: therefore NO must meet the hyperbola. Let it meet it in the point V; and it remains to be proved that it does not meet it in any other point: for, if possible, let it meet the hyperbola likewise in X, and through V, X draw VY, XL parallel to AM; therefore the rectangle NVY is equal to the rectangle NXL; which is absurd: therefore NO meets the hyperbola no where but in the point V. Lastly, in the straight line NV produced, take the point X, and through X draw a straight line parallel to AN, and let this parallel meet AY in L and the hyperbola in Z; therefore the rectangle XLA is greater than VYA, that is, than ZLA; therefore LX is greater than LZ; and thus the point X is within the hyperbola.

Cor. 1. It appears from the demonstration, that a straight line drawn through the centre, and passing between the asymptotes; meets the hyperbola in one point only; for should AR meet the hyperbola in another point O, the rectangles RTA, ONA would be equal; which is absurd.

Cor. 2. And if a straight line meet a hyperbola, or opposite hyperbolas, in two points, it meets both the asymptotes: for if it were parallel to the one of the asymptotes, it would meet the hyperbola in only one point.

Cor. 3. And if a straight line touch a hyperbola, it meets both the asymptotes: for if it were parallel to the one of them, it would pass within the hyperbola; which is absurd.

Cor. 4. If through the point N in one asymptote a straight line NO be drawn parallel to the other, and in this straight line, and within the angle containing the hyperbola, a point V be taken, making the rectangle VNA, contained by a straight line between the asymptote AM and the point V, and the abscissa between it and the centre, equal to the rectangle PMA, contained by a straight line drawn from any point P of the hyperbola.

so as to be parallel to the asymptote AL, and the abscissa between this parallel and the centre; the point V is in the hyperbola. For if NO meet not the hyperbola in V, let it, if possible, meet it in X: the rectangle XNA is, therefore, equal to the rectangle PMA, that is, to the rectangle VNA; which is absurd. Therefore the point V is in the hyperbola.

PROP. XIX. Theor.

Fig. 13. If through a point A of a hyperbola a straight line be drawn meeting both the asymptotes in the points B, C; if from either of the points C, another straight line CD be placed equal to the straight line intercepted between the point A in the hyperbola and the remaining point B, so that the extremity D of the straight line CD, and the point A in the hyperbola, may be either both between, or beyond, the points B, C; the point D, in the first case, is in

the hyperbola in which the point A is; but in the second, it is in the opposite hyperbola.

Let G be the centre of the hyperbolas, and through A, D, to either asymptote GB, draw straight lines AE, DF parallel to the remaining asymptote: and, because of the parallels, BA is to DC, as BE to FG; but BA, DC are equal; therefore BE, FG are equal; and consequently BF, EG are also equal: and because of the equiangular triangles, AE is to DF, as BE to BF, that is, as FG to EG; therefore the rectangle AEG is equal to the rectangle DFG: but the point A is in the hyperbola; and because GF is an asymptote, the point D is also in the hyperbola (4. cor. preced. prop.).

PROP. XX. Theor.

If a straight line cuts the asymptotes, but opposite to the angle adjacent to that containing a hyperbola; it meets each of the hyperbolas in only one point.

BOOK III. Fig. 13. Let there be a hyperbola, the asymptotes of which are GB, GC, and let the straight line BC cut them in the points B, C; and having taken in the hyperbolas any point H, and drawn HIK parallel to BC, meeting the asymptotes in I, K; to the straight line BC apply a rectangle equal to the rectangle IHK, and exceeding by a square (29. 6. Elem.); and let either A, or D be the point of application; the points A, D are in the hyperbolas. For through A, H draw the straight lines AE, AN, and HL, HM parallel to the asymptotes; and because the rectangles BAC, KHI are equal, BA is to KH as HI to AC: but, because of the equiangular triangles, BA is to KH as AE to HL; and HI is to AC, as HM to AN; therefore AE is to HL, as HM to AN; and therefore the rectangle EAN, or AEG, is equal to the rectangle MHL, or HLG: but the point H is in the hyperbola; therefore the point A is also in the same, or in the opposite hyperbola (4. cor. 18. 3.). In the same manner D is shown to be in the hyperbola opposite to that in which the point

A is. And it is manifest, that BC does not meet the hyperbolas in any other point.

Cor. Hence, if a straight line BC cut both the asymptotes, but opposite to the angle adjacent to that containing the hyperbola, and in BC produced a point A be taken such, that the rectangle BAC be equal, either to KHI, contained by the segments of any straight line HK parallel to BC, intercepted between the point H where HK meets the hyperbola, and the points K, I where it meets the asymptotes, or to the square of the semidiameter parallel to BC; the point A is in one of the hyperbolas (1. cor. 15. 3.).

PROP. XXI. Theor.

If a straight line cut both the asymptotes of a hyperbola, and if the square of half this line be not less than the rectangle contained by the segments of another straight line, drawn parallel to it, through any point of the hyperbola, intercepted between the hyperbola and the asymptotes; this straight line meets the hyperbola

Fig. 11.

Let there be a hyperbola, the asymptotes of which are OC, OD, and let a straight line GH cut them; in the hyperbola take any point M, and through M draw a straight line parallel to GH, meeting the asymptote in K, L; if the square of half GH is not less than the rectangle KML, the straight line GH meets the hyperbola.

To the straight line GH apply a rectangle equal to the rectangle KML, and deficient by a square; which, from the determination, is possible (27. 28. 6. Elem.), and let A be the point of application; this point will be in the hyperbola: for if the straight lines AC, MN be drawn through the points A, M parallel to the asymptotes, the rectangles ACO, MNO will be equal (1. cor. 16. 3.); because the rectangle GAH is (15. 3.) equal to the rectangle KML; but the point M is in the hyperbola, therefore the point A is also in it. In like manner it may be proved, that the other point of application is in the hyperbola: but if the square of the half of GH be equal to the rectangle KML, the point bisecting GH is the only point of GH that can be in the hyperbola.

Cor. Hence, if in a straight line GH cutting the asymptotes OK, OL of a hyperbola, a point A be taken, making the rectangle GAH equal either to the rectangle KML, contained by the segments of any other straight line KL parallel to GH, intercepted between the point M where KL meets the hyperbola, and the points K, L, where it meets the asymptotes; or, equal to the square of the segment of the tangent parallel to GH, between the asymptote and point of contact; the point A is in one of the hyperbolas.

PROP. XXII. *(Prop. 14. B. 2. Apoll.)*

An asymptote and the hyberbola, produced indefinitely, continually * approach; and the distance between them becomes less than any given distance.

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* See Proposition XII.

BOOK III. Fig. 14. Let there be a hyperbola, the asymptotes of which are AB, AC, and let D be the given distance; and let E, F be two points in the hyperbola, through which draw GEH, CFL parallel to each other, and meeting the asymptotes in the points G, H and C, L; join AE, and let it meet CL in K; then, because the rectangle GEH (15. 3.) is equal to the rectangle CFL, LF is to HE, as EG is to FC: but LF is greater than HE, because KL is greater than HE; therefore EG is also greater than FC. In like manner it may be proved, that the parallels which follow are successively less than FC. Take then a distance GM less than the given distance D, and through M draw MN parallel to AC; therefore MN will meet (18. 3.) the hyperbola: let it meet it in N, and through N draw ONB parallel to GH; therefore the distance ON is equal to GM, and therefore less than the given distance D.

PLATE X.

Fig. 10. N. 2.

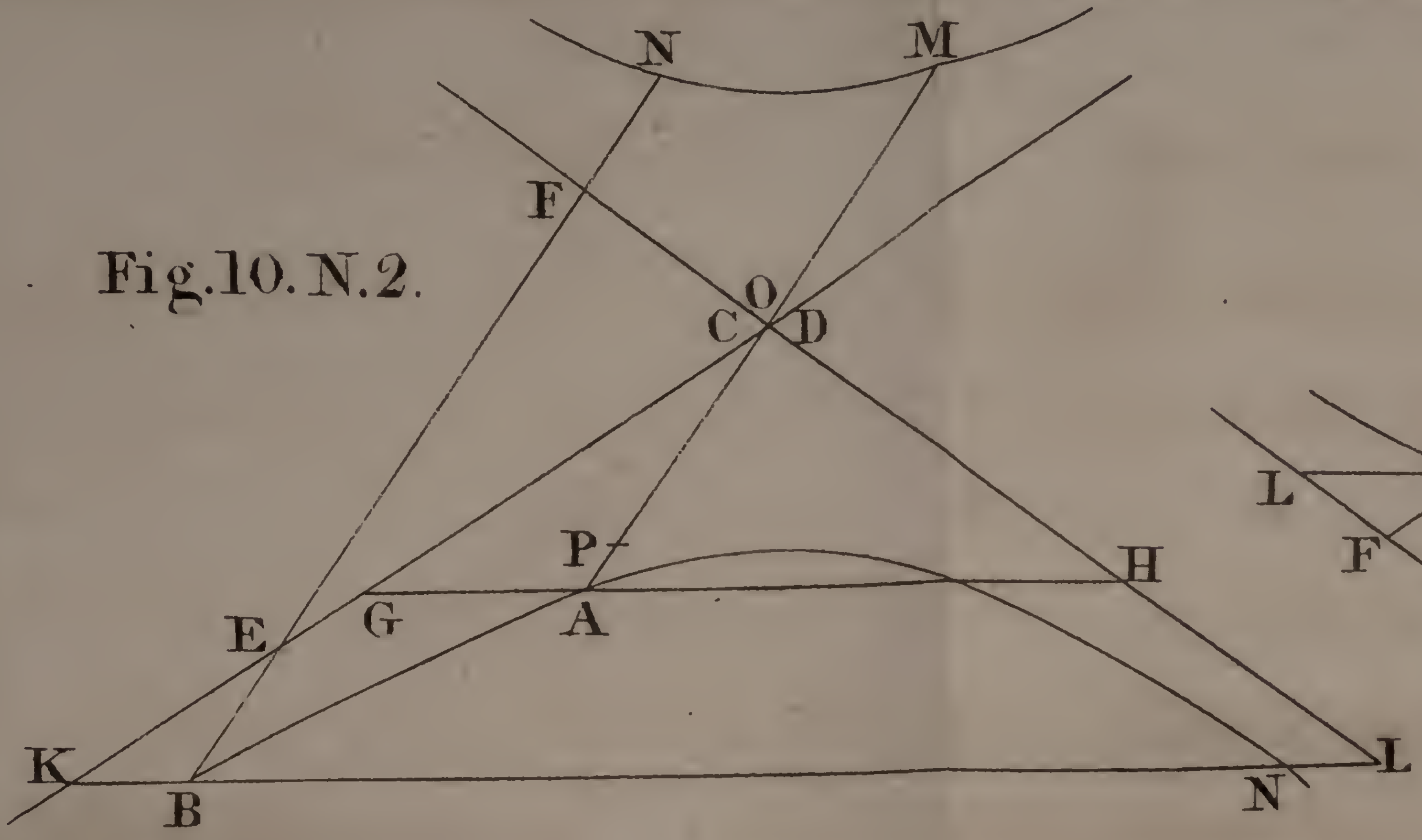


Fig. 11.

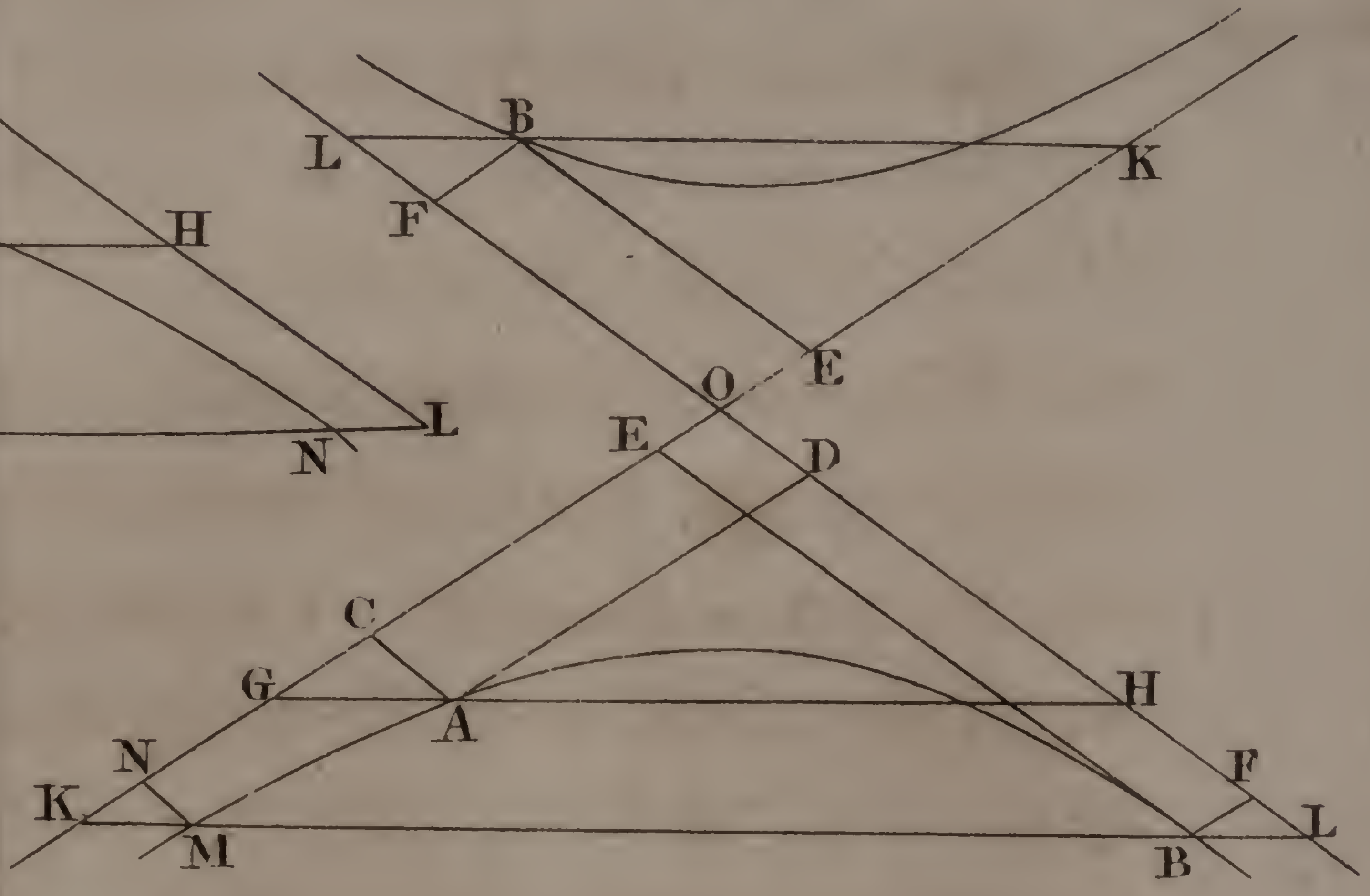


Fig. 12.

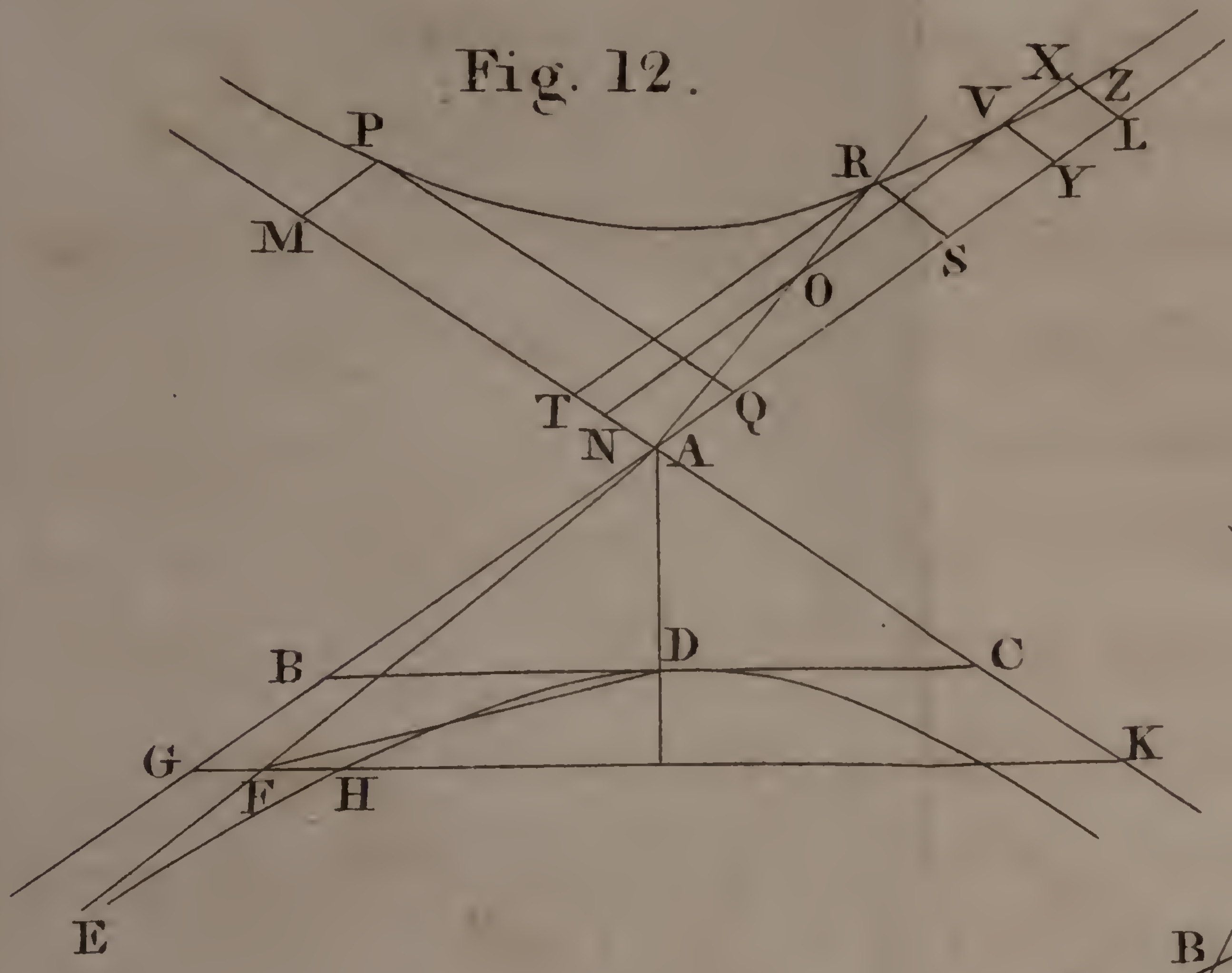


Fig. 13.

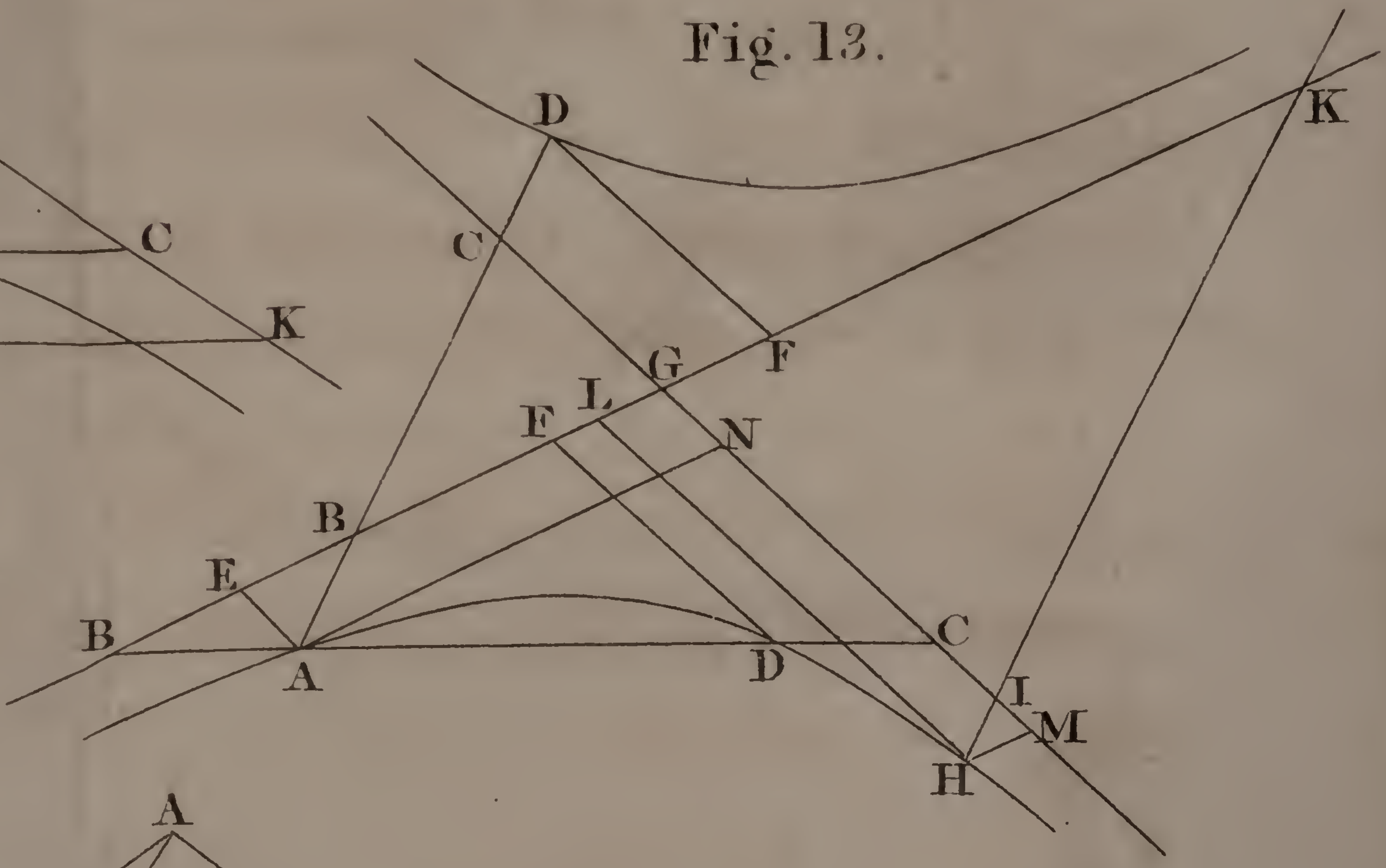
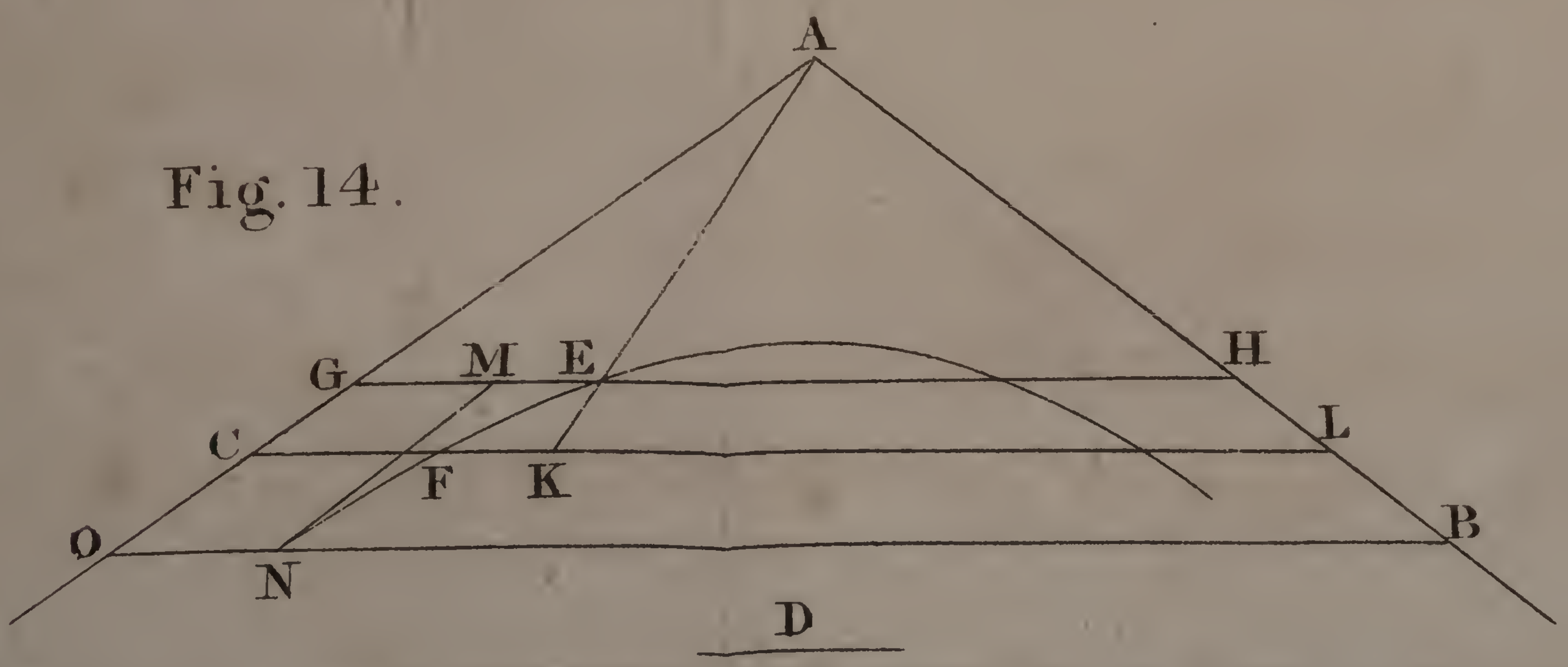


Fig. 14.



PROP. XXIII.

If a straight line intercepted between the asymptotes meets the hyperbola, and is bisected in the point where it meets it; this line touches the hyperbola: and if it touch the hyperbola, it is bisected in the point of contact.

Fig. 15. n. 1.

Let there be a hyperbola, the asymptotes of which are AB, AC, and let a straight line BC, terminated by the asymptotes, meet it in the point D, and be bisected in D; the straight line BC touches the hyperbola.

Through D draw DE parallel to the one asymptote AC, and meeting the other in E; and in BC take any point G, through which draw GH parallel to DE; GH will meet the hyperbola (18. 3.) in some point F: then, because BD, DC are equal, BE, EA are also equal; and, because of the equiangular triangles, BE is to ED, as BH to HG; therefore (1. 6. Elem.) the rectangle BEA is to the rectangle DEA, as the rectangle BHA to

GHA: but the rectangle BEA is (5. 2. Elem.) greater than BHA; therefore the rectangle DEA is also greater (14. 5. Elem.) than the rectangle GHA; that is, because F is in the hyperbola, the rectangle FHA is greater than the rectangle GHA; and therefore FH is greater than HG: therefore the point G is without the hyperbola; and therefore the straight line BC touches the hyperbola in the point D.

OTHERWISE.

Fig. 15. n. 1.

If a straight line LM, terminated by the asymptotes, is bisected by the hyperbola in the point D, it touches the hyperbola in this point.

It is plain that the straight line LD passes not within the hyperbola; for if it passed within the hyperbola it would necessarily meet it again in another point, because the points L, M are without the hyperbola: but it is impossible for it to meet the hyperbola in any other point but D. For, if possible, let it meet it likewise in N; therefore NM is

(14. 3.) equal to DL, that is, according to the hypothesis, to DM: which is absurd. Therefore LM falls not within the hyperbola, nor meets it any where but in the point D; and therefore LM touches it in D.

On the contrary: if the straight line LM, terminated by the asymptotes, touch the hyperbola in D, it is bisected in the point of contact.

For if LD, DM are unequal, from DM the greater take away MN equal to LD the less; therefore the point N is (19. 3.) in the hyperbola; and therefore, contrary to the hypothesis, LM cuts the hyperbola.

Fig. 15. n. 1.

Cor. 1. Hence through the same point of a hyperbola, only one straight line can be drawn touching the hyperbola.

Let D be a point in the hyperbola, and through that point to the asymptote AB draw a straight line DE parallel to the other; and take EB equal to EA, and having joined BD, let it meet the asymptote AC in C: then, since BE, EA are equal, BD, DC are also equal; BC, therefore, touches the hyperbola in D. And no other straight line can touch it in the same point D: for, if possible, let LDM also touch it; then, since BE, EA are equal, therefore LE, EA are unequal; and consequently LD, DM are likewise unequal: therefore LM does not touch the hyperbola.

Cor. 2. Hence is manifest, the manner by which, if the asymptotes AB, AC of a hyperbola be given in position, a straight line BC can be drawn, which shall touch the hyperbola in a given point D.

Fig. 15. n. 2.

Cor. 3. If through the vertices of a transverse diameter two straight lines be drawn touching the hyperbolas, they are parallel to each other. Let AC, BC be the asymptotes, and let AOB, QPR touch the hyperbolas in the vertices of the transverse diameter OCP; the tangents AB, QR are parallel. Draw to either asymptote AC the straight lines OS, PT parallel to the other, and the triangles SCO, TCP are equiangular; by the proposition, AO, OB are equal, and because of the parallels, AS, SC are also equal: and, in like manner, QT, TC are equal; and CO is to CP, as CS to CT, and consequently, as CA to CQ; therefore the triangles OCA, PCQ are equiangular; and therefore OA, PQ are parallel.

Cor. 4. And if a straight line be drawn parallel to a tangent, and meeting the hyperbola; the square of the segment of the tangent between the point of contact and either of the asymptotes, is equal to the rectangle contained by the segments of the parallel, between either point of concourse with the hyperbola and the asymptotes. For this rectangle is equal to the rectangle contained by the segments of the tangent (15. 3.) between the point of contact and the asymptotes, that is, equal to the square of its segment between the point of contact and either of the asymptotes.

PROP. XXIV. Prob.

Fig. 15. n. 1.

The asymptotes AB, AC of a hyperbola, and a point F in the same, being given in position; to draw a straight line which shall touch the hyperbola, and be parallel to a straight line KO, which is given in position, and cuts both the asymptotes of the hyperbola, or opposite hyperbolas.

Suppose the problem solved; and let BC be parallel to KO, and touch the hyperbola in D; and having joined AD, let AD meet KO in P; draw FRQ parallel to AD, and meeting the asymptotes in Q, R; and since the straight line BC touches the hyperbola in D, therefore BD is equal to DC (23. 3.); and consequently KP is equal to PO; and and KO is given in position and magnitude; therefore KP and the point P are given; but the point A is given; therefore the straight line PAD is given in position. Now the square of AD is equal to (1. cor. 15. 3.) the rectangle QFR; and since FRQ is given in position (28. dat.), and that AB, AC are likewise given in position; therefore FQ, FR are (25. 26. dat.) given; and therefore the rectangle QFR is given; consequently the square of AD is given; and therefore AD is given in magnitude: but the point A is given in position; therefore the point D is also given (27. dat.); and (28. dat.) therefore the straight line BDC is given in position.

The composition is thus: let KO be bisected in P; and having joined AP, draw through the point F a straight line FRQ parallel to AP, and meeting the asymptotes in the points R, Q; in AP produced, and in either direction from the centre, take AD a mean proportional between FQ, FR; and and through D draw BDC parallel to KO; then BC will touch the hyperbola in D. For since the square of AD is equal to the rectangle QFR, the point D is (cor. 17. 3.) in the hyperbola; and since KO, BC are parallel, and that KO is bisected in P by the straight line PAD, therefore BC is bisected in D; and consequently touches the hyperbola in the same point D (23. 3.).

PROP. XXV. Theor.

If two straight lines touch a hyperbola, or opposite hyperbolas, and cut the asymptotes; the rectangle contained by the abscissas of the asymptotes between the centre and the one straight line, is equal to the rectangle contained by the ab-

BOOK III. scissas between the centre and the other straight line.

Fig. 16. Let there be a hyperbola, with AB, AD for its asymptotes, let a straight line BD touch it in C, and let another straight line GE touch the same, or the opposite hyperbola, in F; the rectangles BAD, EAG will be equal.

From the points C, F draw CH, CK, and FL, FM parallel to the asymptotes; then because BCD touches the hyperbola, BC is equal to CD, (23. 3.); and consequently BA is the double of AH, and AD the double of HC; therefore the rectangle BAD is the quadruple of the rectangle CHA. It may be shown in the same manner, that the rectangle EAG is the quadruple of the rectangle FMA: but (16. 3.) the rectangle CHA is equal to the rectangle FMA; the rectangle BAD is therefore equal to the rectangle EAG.

PROP. XXVI. THEOR.

If two straight lines touching a hyperbola, or opposite hyperbolas, meet the asymptotes; the straight lines drawn between the point of concourse are parallel to each other, and to the straight line joining the points of contact.

Let there be a hyperbola, with AB, AD for its asymptotes; let BD touch it in C, and EG touch the same, or the opposite hyperbola, in F; join BE, DG, and CF; the straight lines BE, DG, and GF are parallel. Fig. 16.

Since the rectangles BAD, EAG are equal, BA is to EA, as GA to AD; therefore BE, GD are parallel: join DF, and let it meet BE in N: then since DF is to FN, as GF to FE, that is, as (23. 3.) DC to CB; therefore BN, CF are parallel.

COR. Hence, of two straight lines touching a hyperbola, their segments between the asymptotes, are cut proportionally, in the point O where the two straight lines intersect each other; and also in C, F the points of contact.

PROP. XXVII. Theor.

Every straight line drawn through the centre of a hyperbola, and passing within the angle formed by the asymptotes, adjacent to that containing the hyperbola, is a right diameter.

Fig. 15. n. 2. Let there be a hyperbola, of which AC, BC are its asymptotes, and draw any straight line CE through the centre, and within the angle ACD, adjacent to the angle ACB; then is CE a right diameter.

In BC produced take any point D, and through D to CE draw a straight line DF parallel to the asymptote CA; and having made DG equal to DC, join GF, and let GF meet CA in H: then, since GH meets the straight lines CA, CD, which contain the angle adjacent to ACB, it must (20. 3.) meet the hyperbolas: let it meet them in the points

K, L; therefore KH, LG are equal (14. 3.): and because CD, DG are equal, and CH, DF parallel, therefore HF, FG are equal; consequently the whole FK is equal to the whole FL; and therefore CF is (4. def. 3.) a right diameter.

DEF. XI.

A straight line drawn through the centre of a hyperbola, bisected in the centre, and parallel to a straight line which touches the hyperbola, and equal to its segment between the asymptotes, is called the *second diameter* of the diameter drawn through the point of contact.

Cor. 1. Hence every second diameteris a right diameter: for it passes within the angle formed by the asymptotes, adjacent to that containing the hyperbola (3. cor. 18. 3.).

Cor. 2. Hence, the straight lines which join the vertices of a transverse diameter, and of its second diameter, are parallel to the asymptotes.

Fig. 15. n. 2. For let OCP be a transverse diameter, and MCN its second, and AOB a straight line touching the hyperbola in the vertex of the transverse OCP; the straight lines MO, NO are (33. 1. Elem. and 11. def. 3.) parallel to CB, CA.

DEF. XII.

A third proportional to two diameters, one of which is a transverse diameter, and the other its second diameter is called the *latus rectum*, or the *parameter* of that diameter which is the first of the three proportionals.

PROP. XXVIII. THEOR.

If from a point in a hyperbola to a transverse diameter, a straight line be drawn parallel to its second diameter; the square of the transverse is to the square of its second diameter, as the rectangle contained by the segments of the transverse between its vertices and the parallel, is to the square of the parallel.

Fig. 17. n. 1.

Let A*a* be a transverse diameter, B*b* its second diameter, and CG, CF the asymptotes; from a point D in the hyperbola to the transverse A*a* draw DE parallel to B*b*; the square of A*a* is to the square of B*b*, as the rectangle AE*a* is to the square of DE.

Let DE meet the asymptotes in F, G, and draw HAK touching the hyperbola in the vertex A; therefore, by def. 11. of this book, HA is equal and parallel to BC, and, of consequence, parallel to FE; and, because of the equiangular triangles, the square of CE is to the square of EF, as the square of CA to the square of AH, that is, as the same square of CA to the (4. cor. 23. 3.) rectangle FDG; the square of CA is, therefore (19. 5. Elem.) to the square of AH, as the rectangle AE*a* to the square of ED; and therefore the square of A*a* is to the square of B*b*, as the rectangle AE*a* to the square of ED.

Cor. 1. The squares of straight lines drawn from points of a hyperbola, or of the opposite hyperbola, to a transverse diameter, and parallel to its second, are to one another as the rectangles contained by the segments of the transverse, intercepted between its vertices and those parallels; as was shown in the ellipsis (1. cor. 15. 2.)

Fig. 17. n. 1. COR. 2. And on the contrary: if a hyperbola AM, having A*a* for a transverse diameter, and B*b* for its second diameter; and if from a point D to the transverse A*a* a straight line DE be drawn parallel to the second, and meeting the transverse produced in E; and if the square of CA be to the square of CB, as the rectangle AE*a* to the square of ED; the point D is in the hyperbola. For since DE is parallel to BC, and consequently to HK, which touches the hyperbola in the vertex of the transverse diameter; DE will necessarily meet the asymptotes (3. cor. 18. 3.), and of consequence the hyperbola, because the point E is in A*a* produced: if, then, it does not meet the hyperbola in D, let it, if possible, meet it in another point *d*, on the same side of A*a* with the point D; therefore the rectangle AE*a* is to the square of *d*E, as the square of CA to the square of CB, that is, by hypothesis, as the rectangle AE*a* to the square of DE; therefore *d*E, DE are equal; which is absurd. Therefore DE meets not the hyperbola in *d*, nor, as is evident, in any point but D.

COR. 3. Substitute the word *hyperbola* in place of *ellipsis*, and the third corollary of prop. 15. B. 2. becomes also a corollary from this proposition.

PROP. XXIX. THEOR.

If from a point of a hyperbola to a second diameter, a straight line be drawn parallel to its transverse diameter; the square of the second diameter is to the square of its transverse, as the sum of the squares of half the second diameter, and the segment between the centre and the parallel, is to the square of the parallel.

Fig. 17 n. 1.

From D, a point of the hyperbola, to the second diameter B*b*, draw DL parallel to its transverse diameter A*a*; the square of B*b* is to the square of A*a*, as the sum of the squares of CB, CL is to the square of DL.

Through the point D draw DE parallel to BC; and since, by the preceding proposition, the square of CA is to the square of CB, as the rectangle AE*a* to the square of ED; therefore, inversely, and by prop. 12. 5. Elem. the square of CB is to the square of CA, as the sum of the squares of CB, ED to the square of CA, together with the rectangle AE*a*, that is, as the sum of the squares of CB, CL to the square of EC or DL.

COR. 1. If from two points of a hyperbola, or of opposite hyperbolas, to a second diameter, two straight lines be drawn parallel to its transverse diameter; the square of the one parallel is to the square of the other, as the sum of the squares of half the second diameter and the distance between the first parallel and the centre, to the sum of the squares of half the same second diameter, and the distance between the other parallel and the centre.

COR. 2. And on the contrary: if from a point D to a second diameter BC of a hyperbola, a straight line DL be drawn parallel to its transverse CA; and if the square of B*b* have the same ratio to the square of Aa, that the sum of the squares of CB, CL have to the square of DL; the point D is in the hyperbola. For since the straight line DL is parallel to the transverse diameter AC, which falls between the asymptotes, it necessarily meets them opposite to the angle adjacent to that containing the hyperbola, and, of consequence, DL meets both the hyperbolas (20. 3.): and, in the same manner, as in cor. 2. 7. it may be proved, that it meets the hyperbola in D.

Cor. 3. The third corollary of the preceding prop. *mutatis mutandis,* is likewise a corollary here.

PROP. XXX. Theor.

Any straight line terminated both ways by the hyperbola, or opposite hyperbolas, and parallel either to a transverse, or its second diameter, is bisected by the other; or, what is the same thing, a transverse diameter, and its second, are conjugate diameters.

Cor. It is evident, that two diameters cannot be conjugated to the same diameter, whether it be a transverse or a second diameter.

PROP. XXXI. Theor.

Any straight line terminated both ways by a hyperbola, and bisected either by a transverse, or its second diameter, is parallel to the other: and therefore straight lines ordinately applied to either of these diameters are parallel to each other.

Cor. 1. Hence, straight lines parallel either to a transverse, or its second diameter, and which cut off equal segments of the other, between the points where they meet it and the centre, are equal. And equal straight lines if parallel to either diameter, cut off equal segments of the other diameter between the centre and the points where they meet it.

Fig. 17. n. 2. These two propositions, and this first corollary, are demonstrated from the 28th and

29th propositions, in the same manner in which the 9th and 10th propositions were demonstrated from the 7th and 8th.

Cor. 2. If several parallels are terminated both ways by a hyperbola, or hyperbolas, the diameter which bisects the one bisects the rest of them. For that parallel which is bisected, is parallel to the conjugate diameter of that which bisects it; the rest, therefore, are parallel to the same conjugate diameter, and consequently are bisected by the other diameter (30. 3.).

Cor. 3. On the contrary: a straight line which bisects two parallels terminated both ways by a hyperbola, or opposite hyperbolas, is a diameter. For if not, draw a diameter bisecting one of the parallels; this diameter will bisect the other; but, by hypothesis, there is also another straight line which bisects both; which is absurd.

Cor. 4. If a straight line touch a hyperbola, that straight line drawn through the point of contact, which bisects any straight line parallel to the tangent, and terminated both ways by the hyperbola, is a diameter. For a parallel to the tangent is (11. def. 3.)

parallel to the conjugate diameter to that which passes through the point of contact: now, if the straight line drawn through the point of contact, and which bisects the parallel to the tangent be not a diameter, draw a diameter through the point of contact; and this diameter will also (30. 3.) bisect the parallel to the tangent, or to the conjugate diameter: which is absurd.

Cor. 5. Two straight lines terminated both ways by a hyperbola, or by opposite hyperbolas, and not passing through the centre, do not bisect each other. For if they are both terminated by the same hyperbola, or by opposite hyperbolas, draw a diameter through the point where they intersect each other; and then by the proposition, they will be both parallel to the conjugate to this diameter; which is absurd. If indeed one of them be terminated by the hyperbola, and the other drawn between the opposite hyperbolas, it is evident that they cannot bisect each other.

PROP. XXXII. THEOR.

A straight line drawn through the vertex of a transverse diameter, and which is parallel to a straight line ordinately applied to that diameter, touches the hyperbola; and if it touch the hyperbola, it is parallel to straight lines ordinately applied to the transverse diameter drawn through the point of contact.

Let there be a hyperbola, the asymptotes of which are CF, CG; let A*a* be a transverse diameter, and through the vertex A draw HAK parallel to DM ordinately applied to A*a*; the straight line HK touches the hyperbola. Fig. 17. n. 1.

For the straight line DM ordinately applied to the transverse diameter A*a* is (31. 3.) parallel to its second, or conjugate diameter; therefore HAK drawn through the vertex A of A*a*, is parallel to the same conjugate, or second diameter; and therefore it touches the

BOOK III. hyperbola (11. def. 3.). And, conversely: if HK touch the hyperbola, it is parallel to the second diameter (11. def. 3.) of CA: but the ordinate DM is parallel to the same (31. 3.); therefore HK, DM are parallel to each other.

PROP. XXXIII. THEOR.

If a straight line that touches a hyperbola meet a diameter, and if there be drawn from the point of contact a straight line ordinately applied to that diameter; the semidiameter is a mean proportional between its segments intercepted, the one between the centre and the ordinate, and the other between the centre and the tangent.

Case 1. When the tangent meets a transverse diameter.

Fig. 18. Let there be a hyperbola, its asymptotes AG, AH, and let the straight line KCH touch the hyperbola in the point C, and meet a transverse diameter BAO in E; and draw CD from the point of contact C, so as to be

ordinately applied to BAO; then AD, AB, AE are proportionals.

Through the vertex B let GBF be drawn parallel to CD, and let it meet the tangent drawn through C in the point N, and let BM drawn parallel to HC meet AG in M; and let DC meet AH in L, and join KF, GH. Then, because GBF, drawn through the vertex of the diameter, is parallel to the ordinate DC, it (32. 3.) touches the hyperbola: and since the tangents HK, GF are cut proportionally in the points C, B, and N (cor. 26. 3.); therefore CN is to NH, as BN to NG; and because of the parallels, LF is to FH, as MK to KG; and since KF, GH are (26. 3.) parallel, FH is to FA, as KG to KA; therefore, *ex æquo*, LF is to FA, as MK to KA; and, by composition, LA is to FA, as MA to KA; therefore (because of the parallels) DA is to BA, as BA to EA.

Case 2. When the tangent meets a second diameter.

Fig. 19.

Let the straight line CE touch the hyperbola, and meet the second diameter AB in E, and also its transverse, or conjugate diameter KAF in G, and CD, CH being drawn from the point of contact C, so as to be ordinately applied to the conjugate diameters; then AD, AB, AE are proportionals.

For by the preceding case, AH, AF, AG are proportionals; therefore the square of AH is to the square of AF (2. cor. 20. 6. Elem.) as AH is to AG; and, by division, (and 5. 2. Elem.) the rectangle KHF is to the square of AF, as HG to GA: but since CH is ordinately applied to AF, the rectangle KHF is to the square of AF, as the square of CH, or AD to the square of AB; therefore (ex æquali) the square of AD is to the square of AB, as HG to GA, that is, as CH, or AD to AE; and therefore (conv. 2. cor. 20. 6. Elem.) AD, AB, AE are proportionals.

Fig. 18. Cor. 1. Since in the first case, where the tangent and ordinate meet the transverse diameter, AD, AB, AE are proportionals; DO is to DB, as EO to EB, that is, the segments (of the diameter) between its vertices and the ordinate are to each other as the segments of the same between the tangent and the same vertices. The demonstration is similar to that given in the second part of prop. 17. B. 2.

Cor. 2. In the second case, when the ordinate drawn from the point of contact passes through the extremity of the second diameter, the tangent passes through the other extremity of the same diameter. For since the distance between the ordinate and the centre is equal to the half of the second diameter, therefore the distance between the tangent and the centre must be equal to the same semidiameter.

PROP. XXXIV. Theor.

Fig. 18. 19.

If from a point C in a hyperbola a straight line CD be ordinately applied to the diameter AB, and a straight line CE be drawn from the same point; if the semidiameter be a mean proportional between the abscissas of AB, which are cut off towards the centre by these straight lines; the straight line CE touches the hyperbola.

For if CE does not touch the hyperbola let CP touch it; therefore, by the preceding proposition, AD, AB, AP are proportionals; but by the hypothesis, AD, AB, AE are proportionals; which is absurd: CE, therefore, touches the hyperbola.

PROP. XXXV. THEOR.

If a straight line touching a hyperbola meet a transverse diameter, there being drawn from the point of contact a straight line ordinately applied to the same diameter; the rectangle contained by the segments of the diameter intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to the rectangle contained by the segments between the ordinate and the vertices of the diameter: and the rectangle contained by the segments between the tangent and the centre, and between the tangent and the ordinate, is equal to the rectangle contained by the segments between the tangent and the vertices of the diameter. But if the tangent meet a second diameter, there being drawn from the point of contact a straight line ordinately applied to that second diameter; the rectangle contained by the segments between the ordinate and the centre, and between the ordinate and the tangent, is equal to the sum of the squares of the semidiameter and of the segment between the ordinate and the centre: and the rectangle contained by the segments between the tangent and the centre, and between the tangent and the ordinate, is equal to the sum of the squares of the semidiameter and of the segments between the tangent and the centre.

Case 1. Let the straight line which touches the hyperbola in C, meet the transverse diameter BAO in the point E, and let an ordinate drawn through the point of contact to the same diameter meet it in D; then the rec- Fig. 18.

tangle ADE is equal to the rectangle BDO, and the rectangle AED to BEO.

For since AD, AB, AE are proportionals, the rectangle DAE is equal to the square of AB; and these equals being taken from the square of AD, the remaining rectangle ADE is equal to the remaining rectangle BDO (2. and 6. 2. Elem.). Next, from the same equals, viz. the rectangle DAE and the square of AB, take away the common square (3. and 5. 2. Elem.) of AE, and the remaining rectangle AED is equal to the remaining rectangle BEO.

Fig. 18. Case 2. Let the tangent and ordinate drawn to the point C, meet the second diameter in the points E, D: then because the rectangle EAD is equal to the square of AB, add to each of these equals the square of AD, and the rectangle ADE will be equal to (3. 2. Elem.) the sum of the squares of AB, AD. Next, if to the same equals, to wit, the rectangle EAD and the square of AB, the square of AE be added; the rectangle AED will be equal to the sum of the squares of AB, AE.

Fig. 15. N. 1.

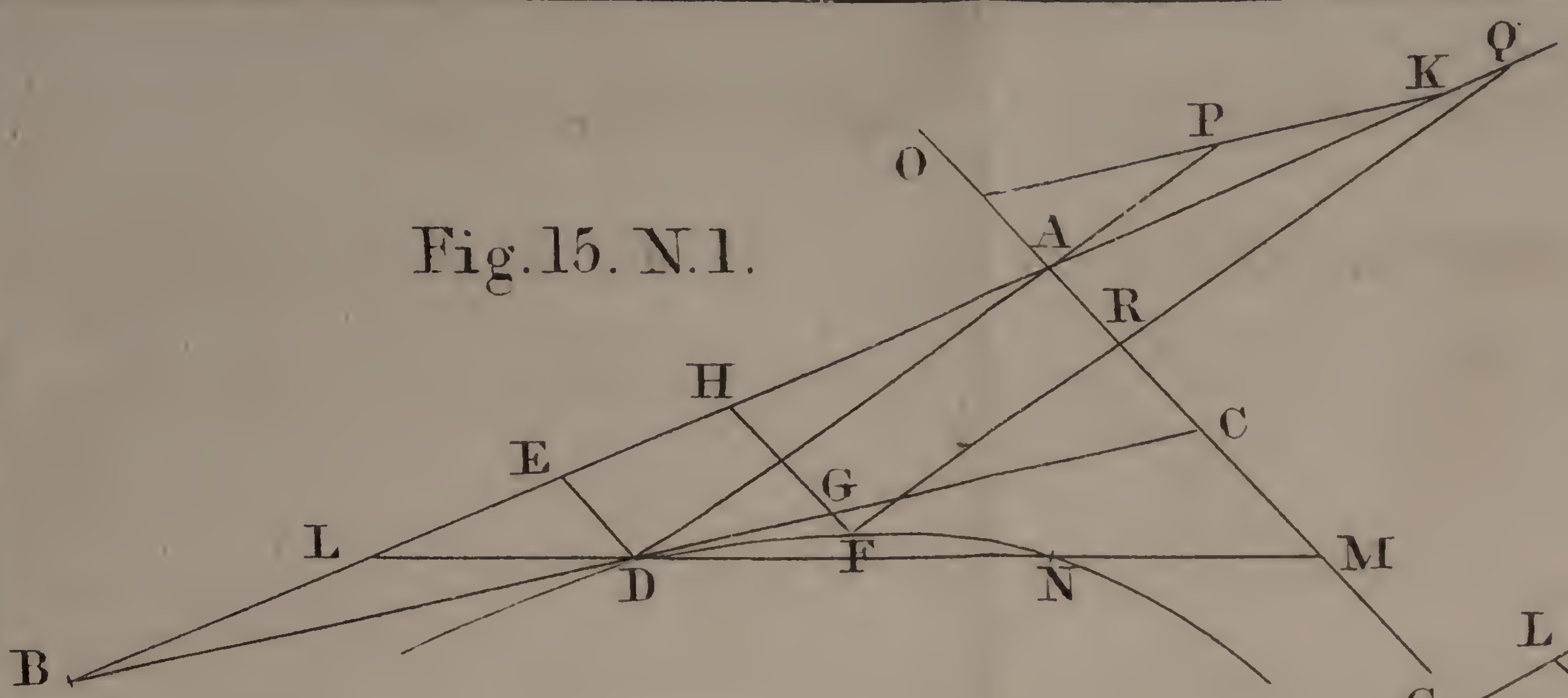


Fig. 16.

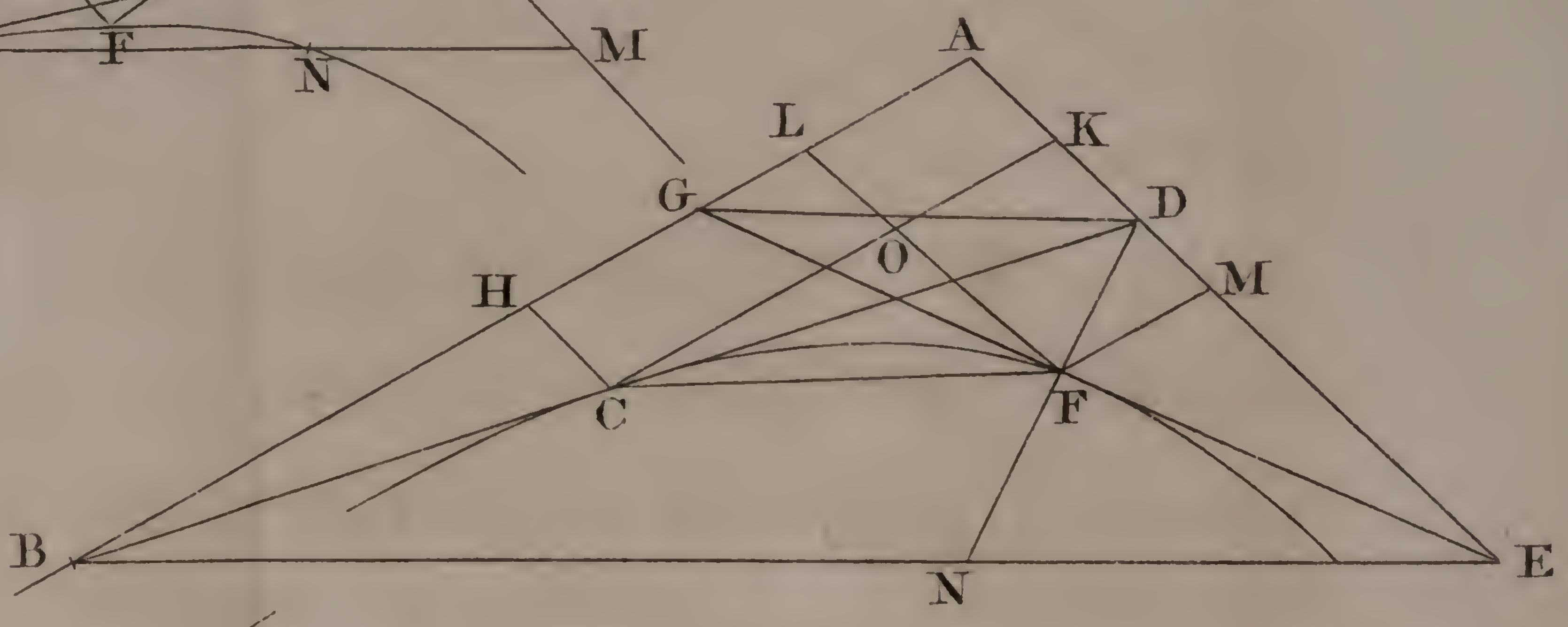


Fig. 15. N. 2.

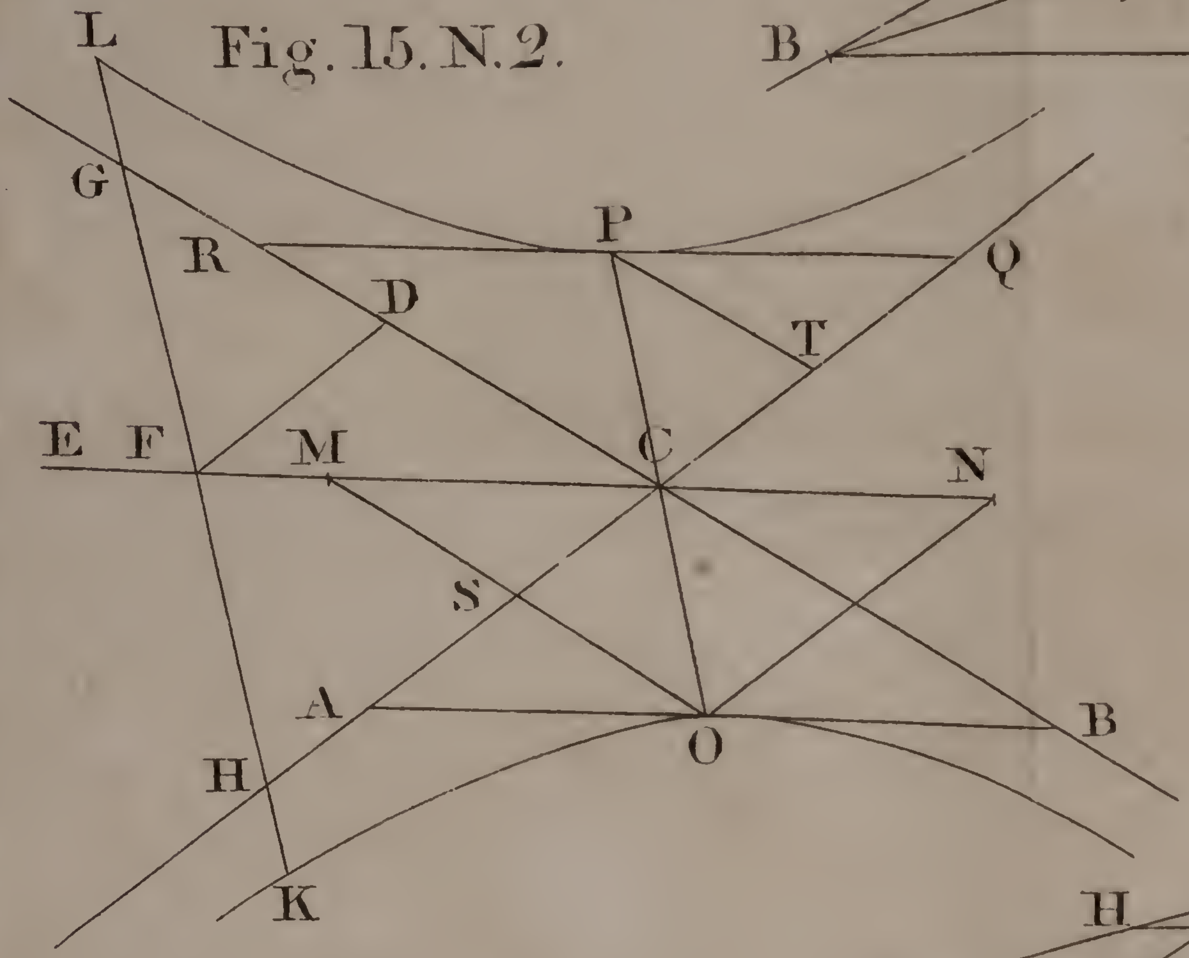


Fig. 17. N. 1.

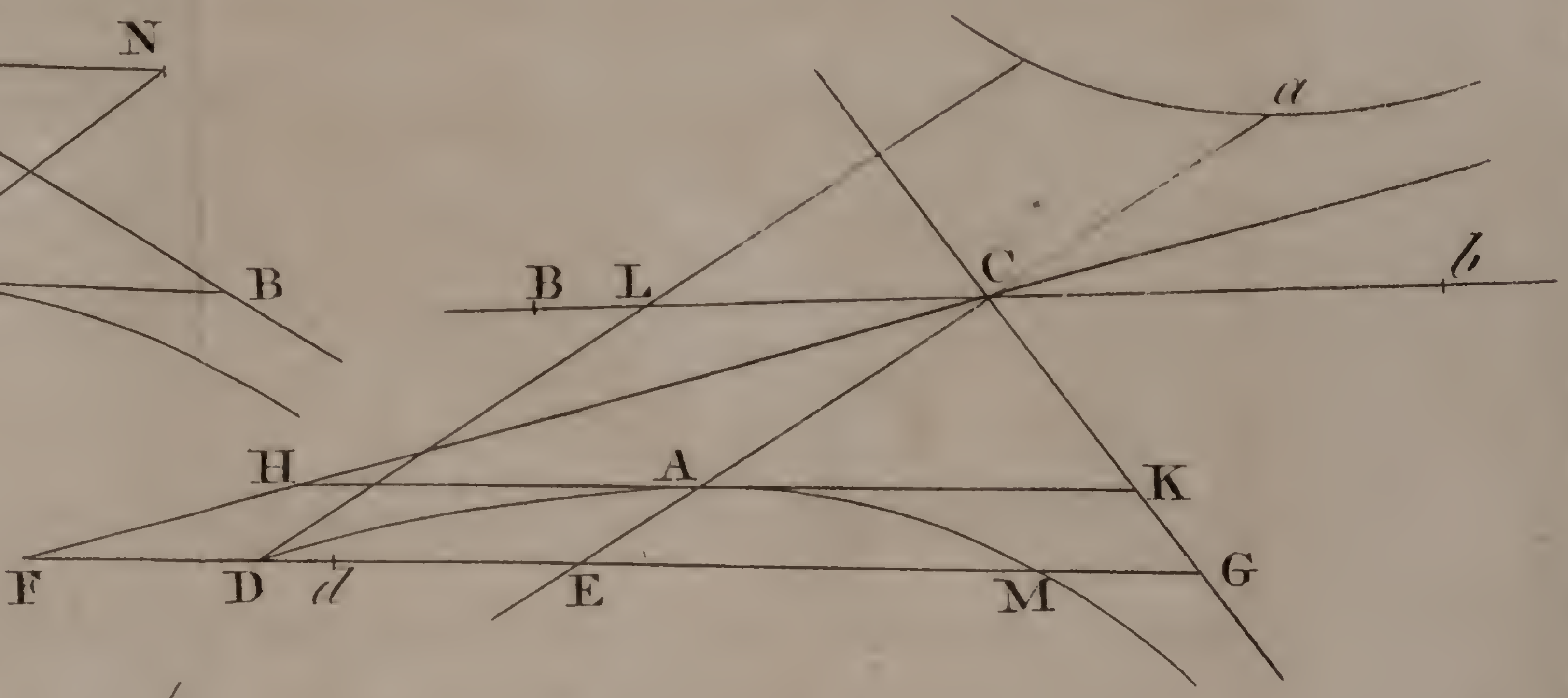


Fig. 17. N. 2.

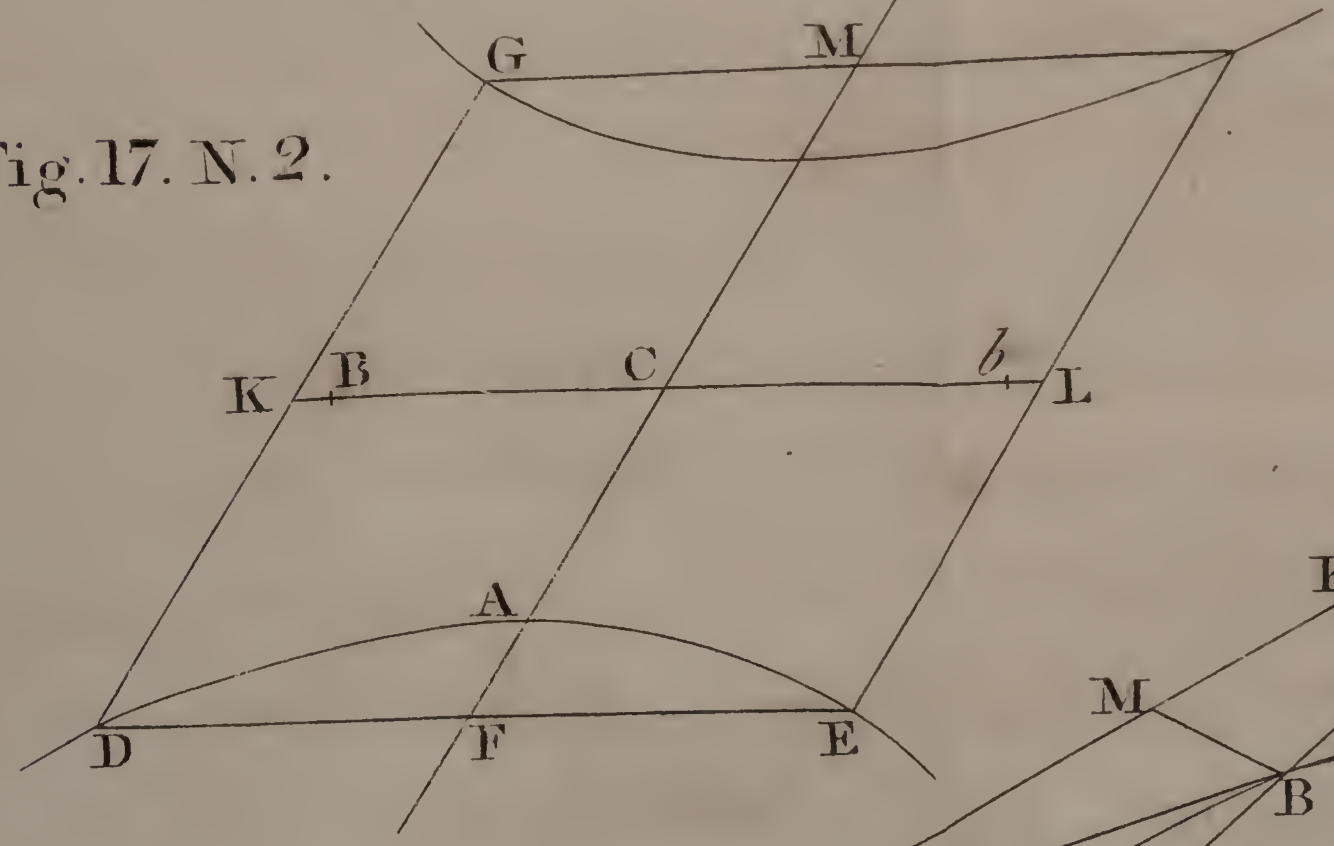
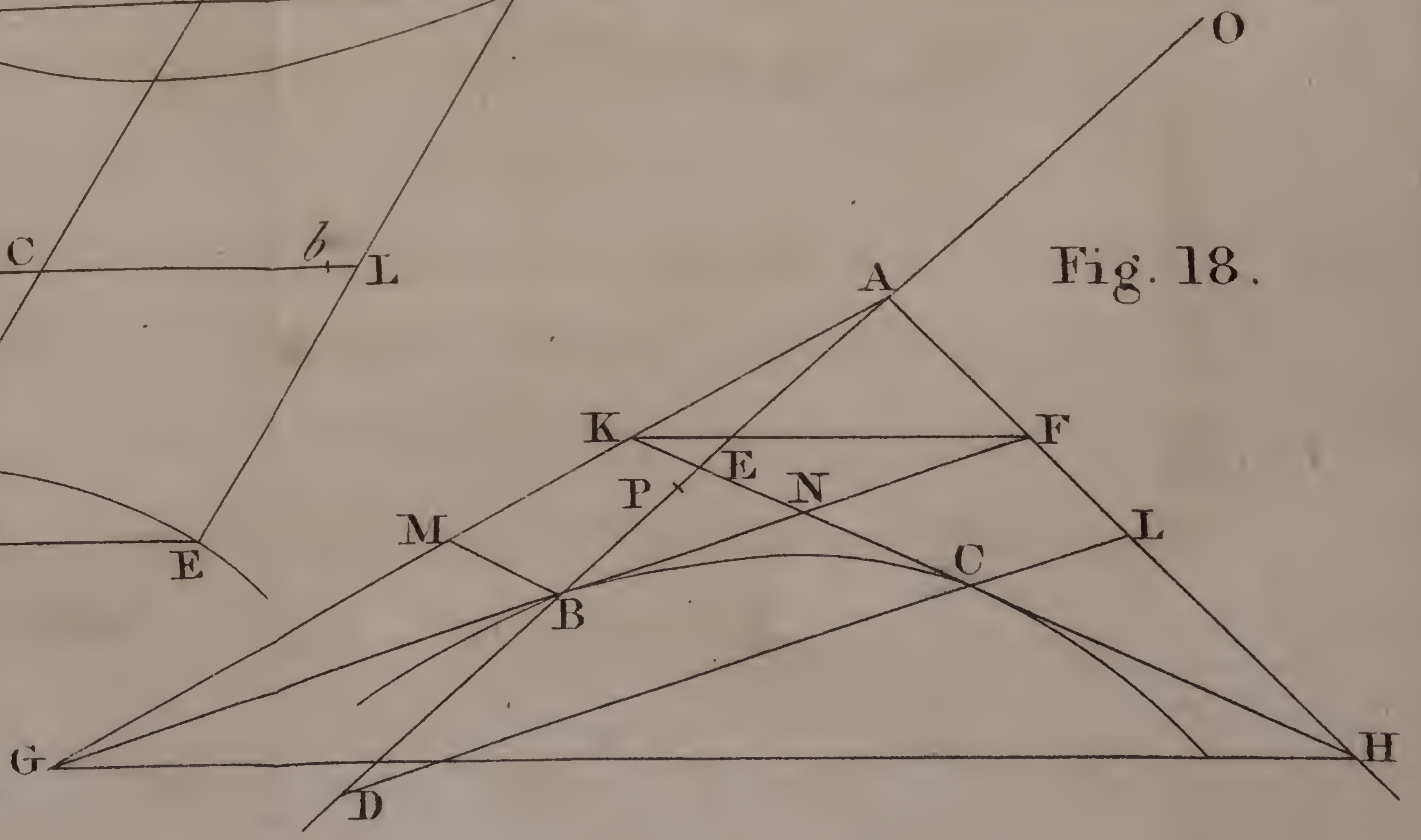


Fig. 18.



PROP. XXXVI. THEOR.

If a straight line touch a hyperbola, it bisects the angle contained by the straight lines drawn from the foci to the point of contact. And, on the contrary: if a straight line bisect the angle contained by two straight lines drawn from a point of a hyperbola to the foci, it touches the hyperbola.

Fig. 20.

Let there be a hyperbola, its transverse axis AB, and the centre the point C, and let a straight line DE touch it, and meet the transverse axis in E, draw the straight lines DF, DG from the point of contact D to the foci; the angles FDE, GDE are equal.

From the point D draw DH perpendicular to the axis, and from the point A, which is the nearer to D of its vertices, place, in the axis produced, a straight line AK equal to DF, and KB will (1. 3.) be equal to DG; and, by the fifth proposition of this book, CK is to CH, as CF to CA; but as CH to CA, so (33. 3.) is CA to CE; therefore, *ex æquali*, as CK to CA, so is CF to CE, and by conversion, as CK is to KA, so is CF to FE; and by doubling the antecedents, twice CK is to KA, as FG to FE: therefore, by division, BK is to KA, as GE to FE; and BK, KA are equal to DG, DF; therefore as DG to DF, so is GE to EF; and therefore (3. 6. Elem.) the straight line DE bisects the angle FDG.

If, on the contrary, a straight line DE bisects the angle FDG, it touches the hyperbola: for if not, let another straight line touch the hyperbola in the point D; this other straight line will bisect the angle FDG, which, by the hypothesis, is bisected by DE; which is absurd. The demonstration here might have been similar to the second demonstration in prop. 11. B. 2.

PROP. XXXVII. PROB.

Fig. 21 Two straight lines AB, CD which bisect each other at right angles in the point E, being given in position and magnitude; to describe the opposite hyperbolas of which they may be the axes, and so that either of them, as AB, may be the transverse axis.

BOOK III.

Join AC, and from the point E place in AB produced two straight lines EF, EG, each of them equal to AC; then, by means of a string and of a ruler, the length of which exceeds that of the string by a difference equal to AB, describe with the foci F, G two opposite hyperbolas; these will pass through the points A, B, and CD will be their second axis.

For if the hyperbola passes not through A, let it pass, if possible, through H; the excess, therefore, of HG above HF is equal to the excess of the length of the ruler above that of the string, that is, by construction, to the straight line AB: but since BG is equal to AF, the excess of AG above AF is equal to the same AB; which is absurd: the hyperbola, therefore, passes through A: and in like manner, it may be shown, that it passes through B. Again, C, D are the extremities of the second axis: for if C be not one of its extremities, let the point K, on the same side

BOOK III. of the centre on which C is, be one of them; therefore KA being joined will be equal (6. def. 3.) to EF; and, by construction, CA is equal to the same EF; therefore KA is equal to CA: which is absurd.

DEF. XIII.

Fig. 22. If upon two straight lines A*a*, B*b*, which bisect each other at right angles, two opposite hyperbolas AG, *ag* be described, and upon the same straight lines other two opposite hyperbolas BK, *bk* be described, so that B*b*, the transverse axis of the latter hyperbolas, may be the second axis of the former, and that A*a*, the second axis of the latter, may be the transverse axis of the former; these four are called *conjugate hyperbolas.*

PROP. XXXVIII. THEOR.

The conjugate hyperbolas have common asymptotes.

Fig. 22. Let there be conjugate hyperbolas, the axes of which are A*a*, B*b*, and let the straight

lines CD, CE be the asymptotes of two opposite hyperbolas, the transverse axis of which is A*a*; the same straight lines are the asymptotes of the two other opposite hyperbolas, the transverse axis of which is B*b*.

Through the vertex A draw DAE parallel to BC, and join DB, and produce it to F; therefore by the tenth definition of this book, BC is equal and parallel to AD or AE; BD is therefore equal and parallel to CA: and because the triangles FBC, CAE are equal and equiangular, BF is equal to CA, that is, to BD; therefore CD, CF are (10. def.) asymptotes of the hyperbola, the transverse axis of which is B*b*, and the second axis A*a*.

PROP. XXXIX. THEOR.

Fig. 22.

If from a point G in one of the conjugate hyperbolas, a straight line GH be drawn parallel to EC, one of the asymptotes, and meeting the other in H; and from the point K in the adjacent hyperbola, a straight line KL be drawn parallel to either asymptote, and meeting the remaining one in L: the rectangles GHC, KLC contained by the parallels, and the abscissas of the asymptotes between the parallels and the centre, are equal. On the contrary: if the point G be in one of the conjugate hyperbolas, and the point K within the angle contained by the asymptotes of of the adjacent hyperbola, the rectangle KLC being at the same time equal to the rectangle GHC; the point K is in the adjacent hyperbola.

Let A*a*, B*b* be the axes of conjugate hyperbolas; join AB, and let it meet the asymptote CD in M, and draw AD parallel to CB: then, since BC, AD are equal and parallel, the triangles CBM, ADM are similar and equal; and consequently AM, MB are equal: the rectangles AMC, BMC are therefore equal, and AB is parallel to the (2. cor. def. 10. 3.) asymptote EC: therefore the rectangles GHC, KLC are equal to the rectangles AMC, BMC (1. cor. 16. 3.); and consequently they are equal to each other.

On the contrary: if G be a point in one of the conjugate hyperbolas, and the point K be within the angle DCF, and the rectangle KLC equal to the rectangle GHC, the construction in other respects still remaining; the point K is in the adjacent hyperbola: for since the rectangle KLC is equal to GHC, that is, to AMC, that is, to BMC, and that B is in the adjacent hyperbola: the point K is (4. cor. 18. 3.) in the same hyperbola.

Fig. 23.

Cor. 1. If a straight line DE, intercepted between a hyperbola and one of the adjacent hyperbolas, is bisected by one of the asymptotes, it is parallel to the other: for let DE meet the asymptote CG in L, and to the other asymptote draw DH, EK parallel to CG; then, since DL, LE are equal, and that DH, LC, EK are parallel, HC, CK are equal; and by the proposition, the rectangles DHC, EKC are equal; DH, EK are therefore equal, and they are parallel; therefore DE, KH are parallel.

Cor. 2. And if DE be parallel to the asymptote KH; DL, LE are equal: for by the proposition, the rectangles DLC, ELC are equal.

Cor. 3. Lastly, if DE be parallel to the asymptote HK, and DL, LE being equal, and the point D be in one of the hyperbolas; the point E is in the adjacent hyperbola: for since DL, LE are equal, the rectangles DLC, ELC are equal; therefore, by the proposition, E is in the adjacent hyperbola BE.

PROP. XL. *(Prop. 20. B. 2. Apoll.)*

If a straight line touch one of four conjugate hyperbolas, and through their centre two straight lines be drawn, the one meeting the hyperbola in the point of contact, and the other parallel to the tangent, and meeting one of the adjacent hyperbolas; this other straight line drawn parallel to the tangent, is the half of the second diameter conjugate to the transverse diameter drawn through the point of contact. And on the contrary: if half a second diameter be drawn conjugate to the transverse diameter passing through the point of contact, its extremity is in the adjacent hyperbola.

Fig. 23.

Let there be conjugate hyperbolas, the asymptotes of which are CG, CF, and let MF touch one of the hyperbolas in D; join CD, and draw CE parallel to MF, and meeting the hyperbola adjacent to that in which D is, in the point E; then is CE half the second diameter conjugate to CD.

Through the points D, E draw the straight lines DH, EK parallel to the asymptote CG, and meeting the other asymptote in H, K: then, since the straight line MF touches the hyperbola in D, MD, DF are equal; therefore CH, HF are equal; therefore the rectangle DHF is equal to the rectangle DHC, that is, by the preceding proposition, to the rectangle EKC: and because the triangles EKC, DHF are equiangular, EK is to DH, as KC to HF; therefore (22. 6. Elem.) the square of EK is to the square of DH, as the rectangle EKC to the rectangle DHF: but the rectangles EKC, DHF, as has been proved, are equal; therefore the squares of EK, DH are equal; and therefore EK, DH are likewise equal; consequently EC, DF are (26. 1. Elem.) equal, and they are parallel; therefore CE is half (11. def. 3.) the second diameter conjugate to CD.

On the contrary: the same construction remaining, if CE be half the second diameter conjugate to CD, its extremity E is in the hyperbola adjacent to that where the point D is: for CE is equal and (11. def. 3.) parallel to DF, and EK is parallel to DH; therefore the triangles EKC, DHF are (26. 1. Elem.) equal; consequently KC is equal to HF, or HC, and EK to DH: and for this reason, the rectangle EKC is equal to the rectangle DHC, and the point D is in the hyperbola, and the point E is within the angle adjacent to that containing this hyper- Fig. 23.

bola; therefore the point E is in the adjacent hyperbola (by part 2. of the preceding).

Cor. 1. If CD be half a transverse diameter, and CE half the second diameter conjugate to CD in the hyperbola AD; on the contrary, CE is a transverse, and CD a second diameter conjugate to CE in the adjacent hyperbola BE.

Join DE and ME, and let ME meet the other asymptote CF in P: then, since DE is (2. cor. def. 11. 3.) parallel to CF, and that MF is bisected in D, therefore MP is bisected in E, and the point E is in the adjacent hyperbola; therefore MP touches that hyperbola: and CD is equal and parallel to ME; for CE, MD are equal and parallel; therefore CD is the second diameter conjugate to the transverse diameter CE in the (11. def.) hyperbola BE.

Cor. 2. The same construction still remaining, the straight line which joins the centre C, and the point of concourse M of the tangents DM, EM, drawn through the vertices of the conjugate diameters, is an asymptote: for if CM be not an asymptote, let CQ, meeting DM in Q, be one; CE, therefore, is equal to DQ: but the same CE is equal to DM; because DMEC is a (11. def.) parallelogram; DQ and DM are therefore equal: which is absurd.

PROP. XLI. THEOR.

If from the extremity of a second diameter of a hyperbola, a straight line be drawn parallel to any transverse diameter, and meeting the second diameter conjugate to this transverse; the square of the parallel, is to the rectangle contained by the segments of the second diameter intercepted between the parallel and its vertices, as the square of the transverse is to the square of the second diameter conjugate to it: and if from the extremity of a second diameter a straight line be drawn parallel to any other second diameter, and meeting the transverse diameter conjugate to this

PLATE XII.

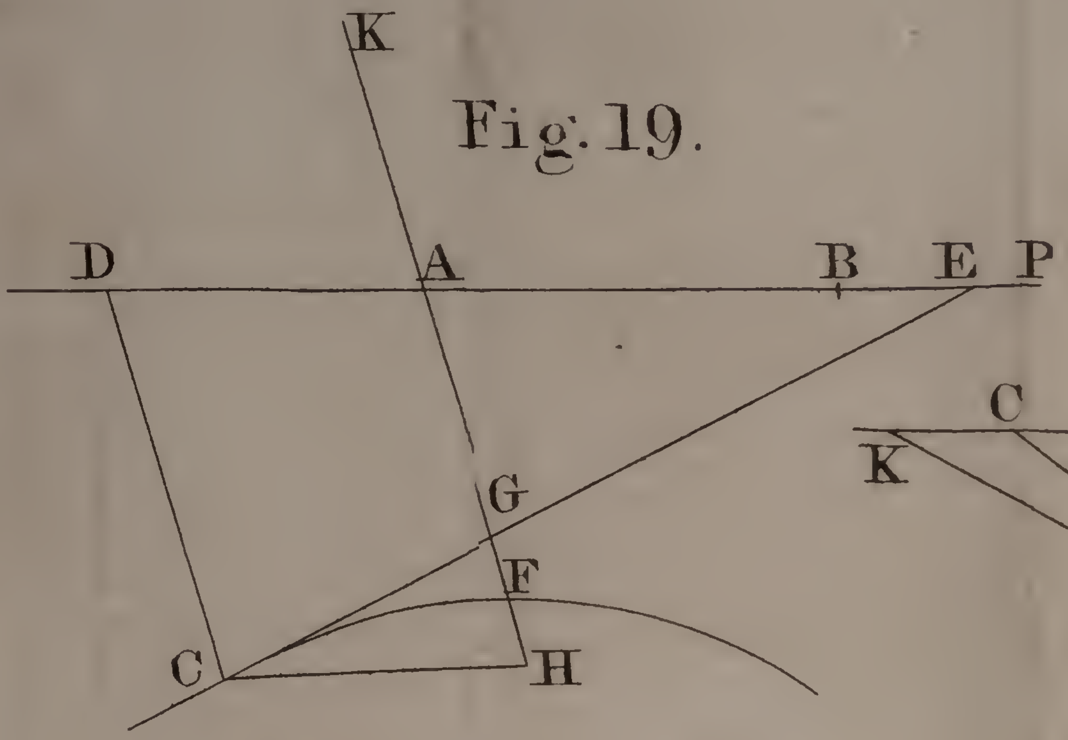


Fig. 19.

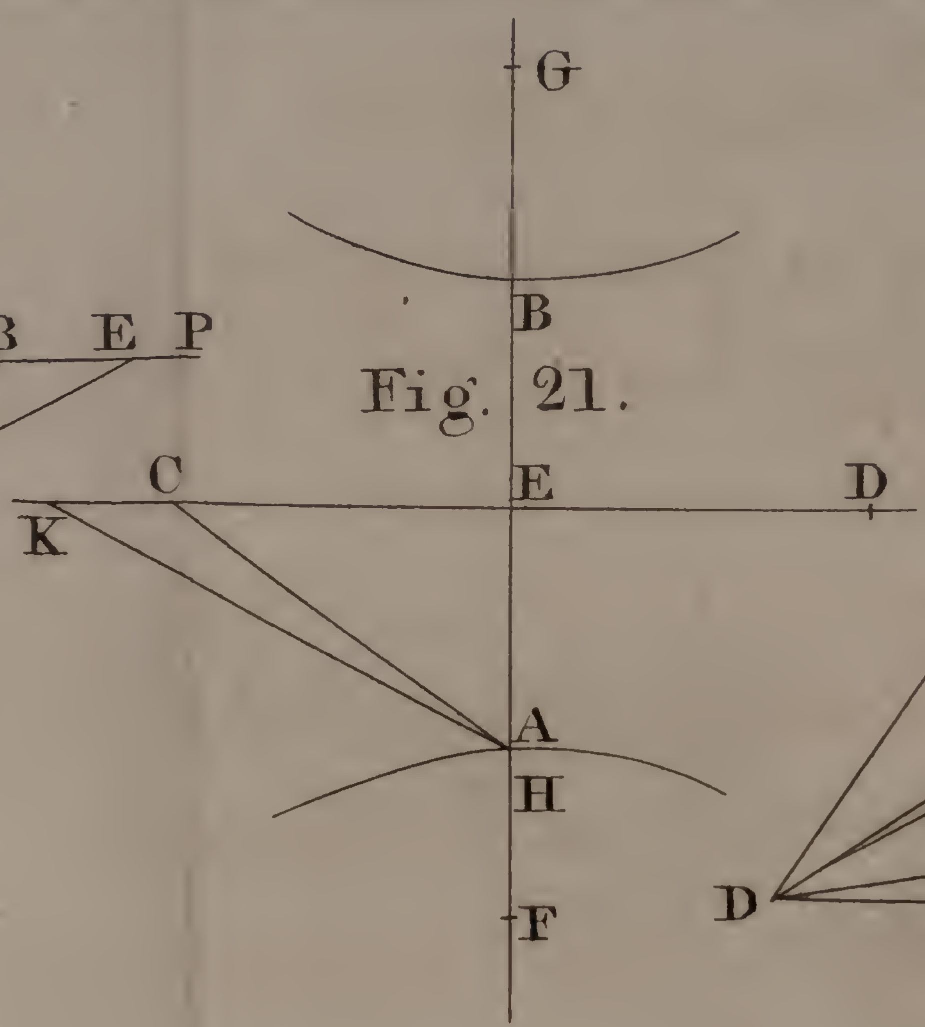


Fig. 21.

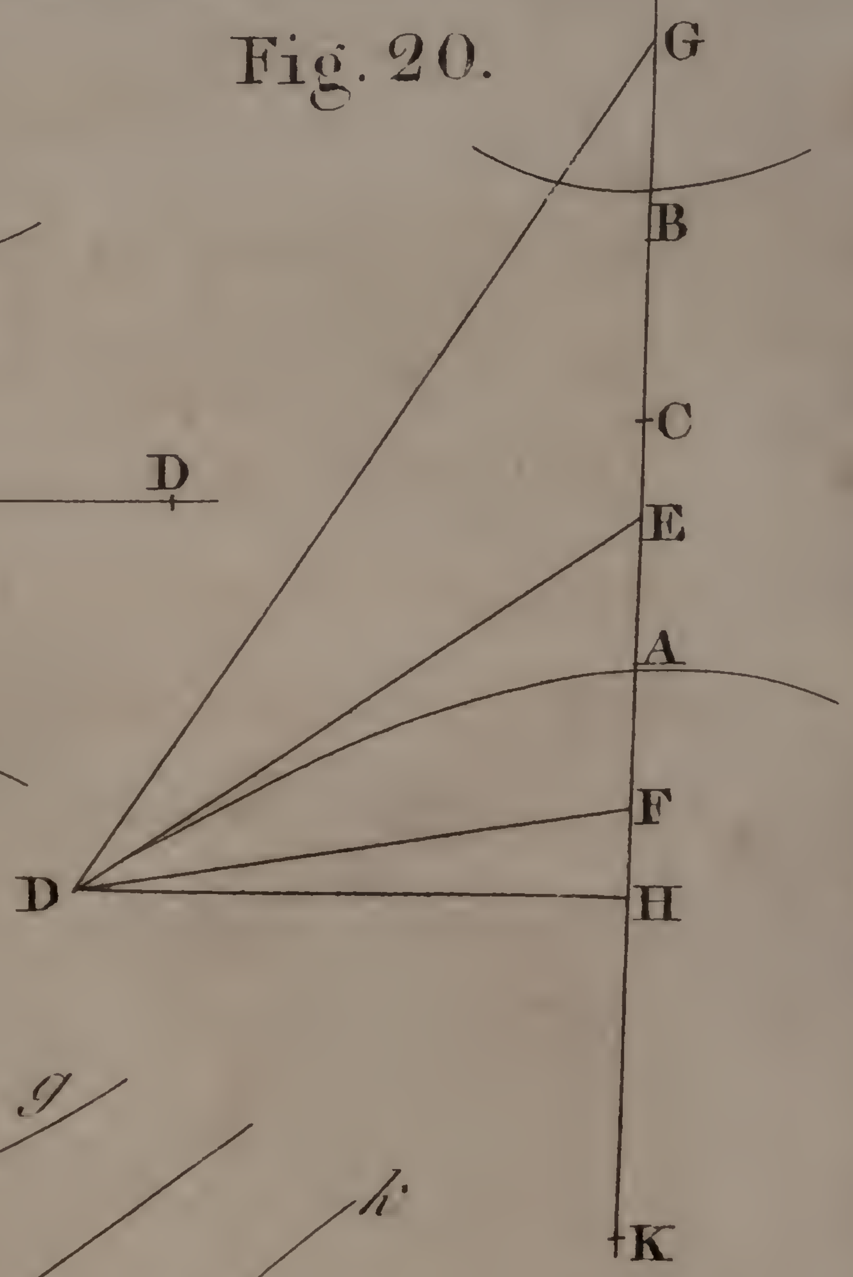


Fig. 20.

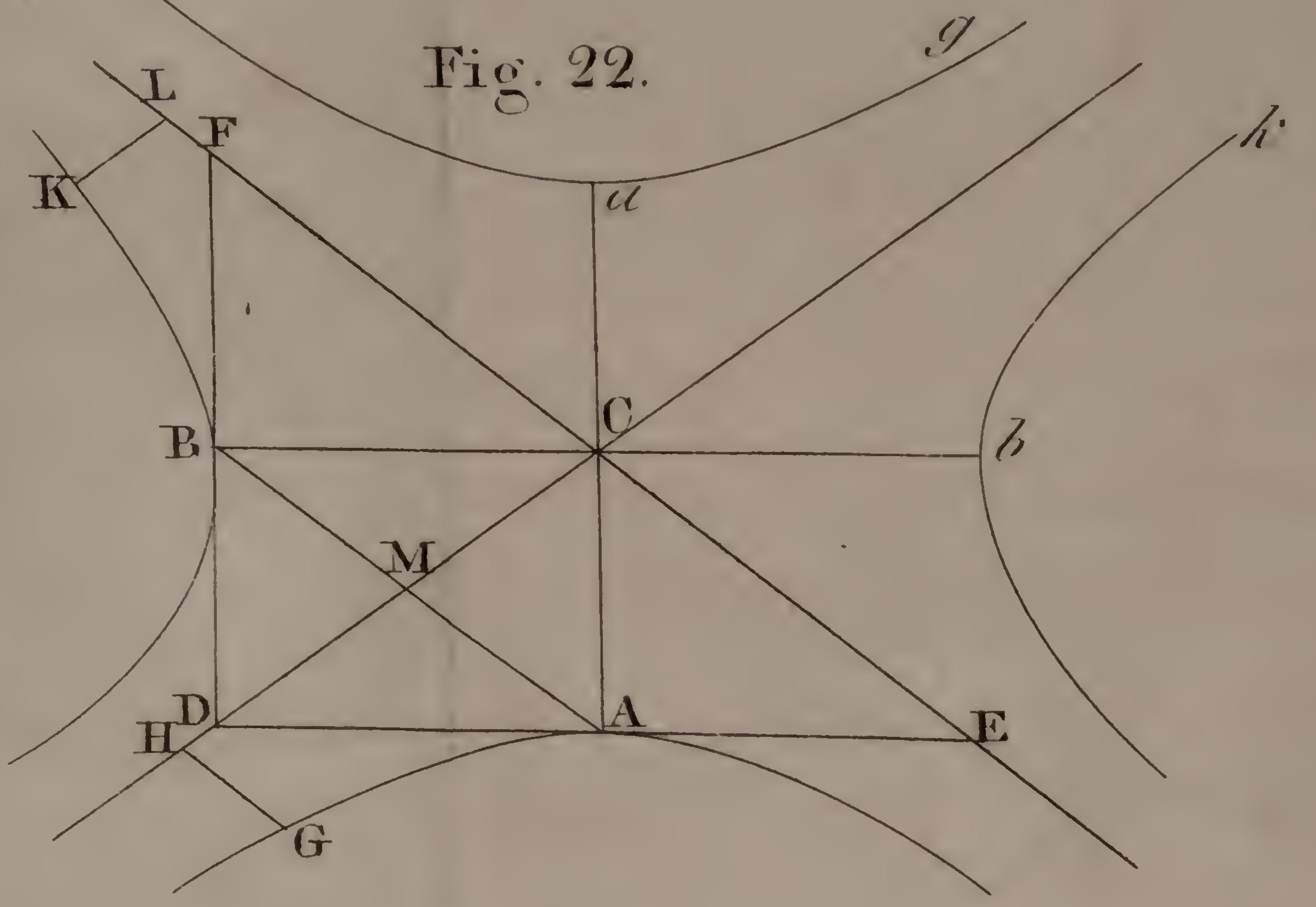


Fig. 22.

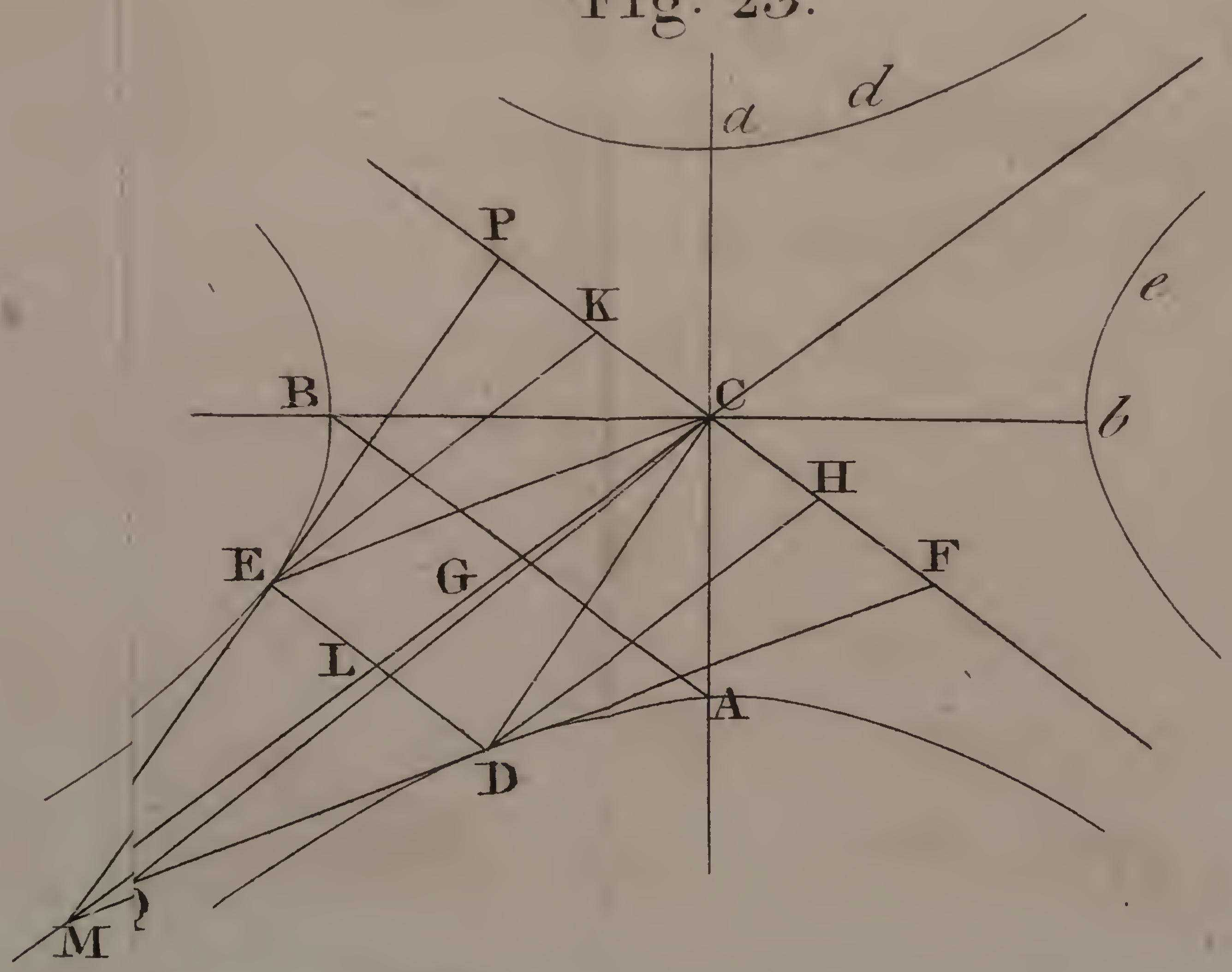


Fig. 23.

other second diameter; the square of the parallel, is to the sum of the squares of half the transverse diameter and its segment intercepted between the centre and the parallel, as the square of that other second diameter, is to the square of the transverse conjugate to it.

Fig. 24.

In the first case let there be a hyperbola, the transverse diameter of which is DC*d*, and let KC*k* be the second diameter conjugate to DC*d*, and let CB be any other second diameter, and from its extremity B draw a straight line parallel to the transverse CD, and meeting in L the second diameter conjugate to CD; the square of BL is to the rectangle KL*k*, as the square of D*d* to the square of K*k*.

For the points B, K, *k*, which are the extremities or vertices of the second diameters, are in the adjacent hyperbolas, of which the transverse diameter K*k* is conjugate to the second (1. cor. 40. 3.) D*d*; therefore the square of BL is to the rectangle KL*k* as the square of D*d* to the square of K*k* (28. 3.).

The second case is demonstrated in the very same manner from the 29th prop. of this book.

Cor. Hence, if from any point A of a hyperbola AD, a straight line AM be drawn ordinately applied to the right diameter K*k*, and from B, an extremity of any second diameter CB to the same K*k*, a straight line BL be drawn parallel to AM; the square of BL is to the square AM, as the rectangle KL*k* is to the sum of the squares of the semidiameter KC and the segment CM between the centre and the ordinate. This is evident from the proposition, and from the 29th of this book.

But if from any point A of a hyperbola AD, a straight line AE be drawn ordinately applied to the transverse diameter D*d*, and from the extremity B of the second diameter to the same D*d*, a straight line BF be drawn parallel to AE; the square of BF is to the square of AE, as the sum of the squares of the semidiameter CD, and the segment CF, between the centre and BF, is to the rectangle DE*d*. This is demonstrated from the proposition, and from the 28th of this book.

PROP. XLII. Theor.

Fig. 24

If from the extremity B of a second diameter CB, a straight line BF be drawn parallel to straight lines ordinately applied to any diameter D*d*, and BH be drawn parallel to the diameter CA, which is conjugate to CB, and meeting the diameter D*d* in H; the semidiameter CD is a mean proportional between CF and CH.

For the extremity B of the second diameter is in the adjacent hyperbola, and CA is a (1. cor. 40. 3.) second diameter of that adjacent hyperbola; therefore BH touches the same (11. def.); and therefore CF, CD, CH are (33. 3.) proportionals.

Cor. The 35th proposition is equally true when accommodated to this case.

PROP. XLIII. THEOR.

If from the extremities of two conjugate diameters of a hyperbola, straight lines be drawn ordinately applied to any third transverse diameter; the square of the segment of the third diameter, intercepted between the centre and the ordinate drawn from the extremity of the transverse diameter, is equal to the square of the segment of the same third diameter between the centre and the ordinate drawn from the extremity of the other of the conjugate diameters, together with the square of half the third diameter. But the square of the segment of the third diameter, intercepted between the centre and the ordinate drawn from the extremity of the second diameter, is equal to the rectangle contained by the segments of the same third diameter, between the ordinate drawn from the extremity of the other of the conjugate diameters, and the vertices of this third diameter.

Fig. 24.

Let there be opposite hyperbolas, in which CA is the half of a transverse diameter, and let CB be the second diameter conjugate to CA, and let D*d* be any other transverse diameter, and from the extremities A, B draw AE, BF ordinately applied to D*d*; the square of EC is equal to the square of FC together with the square of CD: and the square of FC is equal to the rectangle DE*d*.

To the diameter D*d* draw AG parallel to BC, and BH parallel to AC; therefore, because of the parallels, the triangles CAG, HBC are equiangular; and since AE, BF (31. 3.) are parallel, CAE, HBF are also equiangular; consequently CE is to HF, as CA to BH, that is, as CG to CH: and since CD is a mean proportional both between CE and CG, and between CF and CH (33. and 42. 3.), CF is to CE, as CG to CH; and therefore CF is to CE, as CE to HF; consequently the square of CE is equal to the rectangle CFH: but the rectangle CFH (cor. 42. 3.) is equal to the sum of the same squares of CF, CD: take the square of CD from each of these equals, and there will remain the rectangle DE*d* equal to the square of CF.

Cor. Hence, the semidiameter CD, to which the ordinates are drawn, is to the conjugate semidiameter CK, as the distance between the one ordinate and the centre is to the remaining ordinate. For the square of CD is to the square of CK, as the rectangle DE*d* is to the square of EA, that is, according to the proposition, as the square of CF is to the square of EA; and therefore CD is to CK, as CF to EA. Again, because the square of CD is to the square of CK, as the sum of the squares of CF, CD to the square of BF (41. 3.), that is, by the proposition, as the square of CE to the square of BF; therefore CD is to CK, as CE to BF.

PROP. XLIV. Theor.

The excess of the squares of any conjugate semidiameters is equal to the excess of the squares of the halves of the axes, if the conjugate diameters be unequal. And if any one diameter be equal to its conjugate, any other diameter is also equal to its conjugate; and in this case, the angle contained by the asymptotes is a right angle.

Let CA, CB be conjugate semidiameters, and CD, CK the halves of the axes, and from A, B draw the straight lines AE, AM and BF, BL ordinates to the axes. Then the excess of the squares of CA, CB is equal to the excess, by which the sum of the squares of CE, EA differs from the sum of the squares of CL, LB: but, by the preceding, the square of CE is equal to the sum of the squares of CF, CD; and, by the same proposition, the square of CL is equal to the sum of the squares Fig. 24

BOOK III. of CM, CK; therefore the excess of the squares of CA, CB is equal to the excess by which the sum of the three squares of CF, CD, EA differs from the sum of the three squares of CM, CK, LB; and the squares of CF, LB are equal; as also the squares of EA, CM: therefore, if these equals be taken away, the excess by which the sum of the three first squares differs from the sum of the other three, is equal to the excess by which the square of CD differs from the square of CK; and therefore the excess of the squares of CA, CB is equal to this same excess.

Fig. 25. Otherwise: let AB, AC be the halves of any two transverse diameters in a hyperbola, AD, AE the asymptotes; and draw the straight lines BD, CE touching it in the points B, C, and meeting the asymptotes in D, E; therefore, by the 11th def. and prop. 30. of this book, BD is equal to half the second diameter conjugate to AB; and CE, in like manner, is equal to half the second diameter conjugate to AC: it is to be proved, that the excess of the squares of AB, BD is equal to the excess of the squares of AC, CE.

Through the points B, C draw BF, CH parallel to the asymptotes, and BG, CK perpendicular to them: therefore the rectangles AFB, AHC are equal (1. cor. 16. 3.); and, of consequence, AF is to AH, as HC to FB, that is, since the triangles are equiangular, as HK to FG; consequently the rectangles AFG, AHK are equal, and their quadruples are equal: and since, through the point of contact B, a straight line BF is drawn parallel to the asymptote, DF, FA are equal; consequently DG is equal to AF, together with FG: and hence four (8. 2. Elem.) times the rectangle AFG is equal to the excess of the squares of DG, GA, that is, since the triangles DGB, AGB are right angled, to the excess of the squares of DB, BA. It may in the same manner be shown, that four times the rectangle AHK is equal to the excess of the squares of EC, CA; and four times the rectangle AFG, as hath been proved, is equal to four times the rectangle AHK; consequently the excess of the squares of DB, BA is equal to the excess of the squares of EC, CA.

But if in a hyperbola any transverse diameter AB is equal to the second diameter BD conjugate to it, any other transverse diameter in the same hyperbola is also equal to its conjugate second diameter, and the angle contained by the asymptotes is a right angle: for since DB, BA, and DF, FA are equal, and BF common, in the triangles DBF, ABF; the angle BFD, and of consequence the angle EAD is a right angle: and because EAD is a right angle, the angle CHE is also a right angle; and EH, HA are equal, and CH common; therefore EC, CA are equal.

PROP. XLV. THEOR.

If through the vertices of two conjugate diameters, four straight lines be drawn touching conjugate hyperbolas, the parallelogram formed by them is equal to that formed by the tangents drawn through the vertices of any other two conjugate diameters.

Fig. 26.

Let AB, CD be conjugate diameters, and through their vertices draw tangents, meeting each other in K, L, M, N; and let EF, GH be any other conjugate diameters, and through the vertices of these drawn tangents, meeting each other in O, P, Q, R; the figures KLMN, OPQR are parallelograms, and equal to each other.

Let S be the centre of the hyperbola; and since both KN, LM, and KL, MN are (3. cor. 23. 3.) parallel, the figure KLMN is a parallelogram. For a like reason, OPQR is a parallelogram: and since AK, CK touch the hyperbolas in the vertices of conjugate diameters, the point K where they meet is in an asymptote. In like manner it may be shown, that the rest of the angles of the parallelograms are in the asymptotes; therefore the asymptotes are the diagonals of the parallelograms; consequently the parallelogram KLMN is the quadruple of the triangle KSN, and the parallelogram OPQR the quadruple of the triangle OSR: but the triangles KSN, OSR are equal, because the rectangles KSN, OSR are equal (25th of this book, and 15. 6. Elem.); therefore the parallelograms KLMN, OPQR are also equal. This proposition might also have been demonstrated like prop. 20. B. 2.

PROP. XLVI. THEOR.

If two conjugate diameters of a hyperbola meet a straight line touching the hyperbola, the rectangle contained by the segment of the tangent intercepted between the point of contact and the conjugate diameters, is equal to the square of the semidiameter conjugate to that diameter which passes through the point of contact.

Fig. 27. Let ACB, DCE be two conjugate diameters, and let a straight line which touches the hyperbola in F meet them in the points G, H, and let CK be the semidiameter conjugate to CF; the rectangle GFH is equal to the square of CK.

From the points F, K draw to AB the straight lines FM, KL parallel to DE: then,

PLATE XIII.

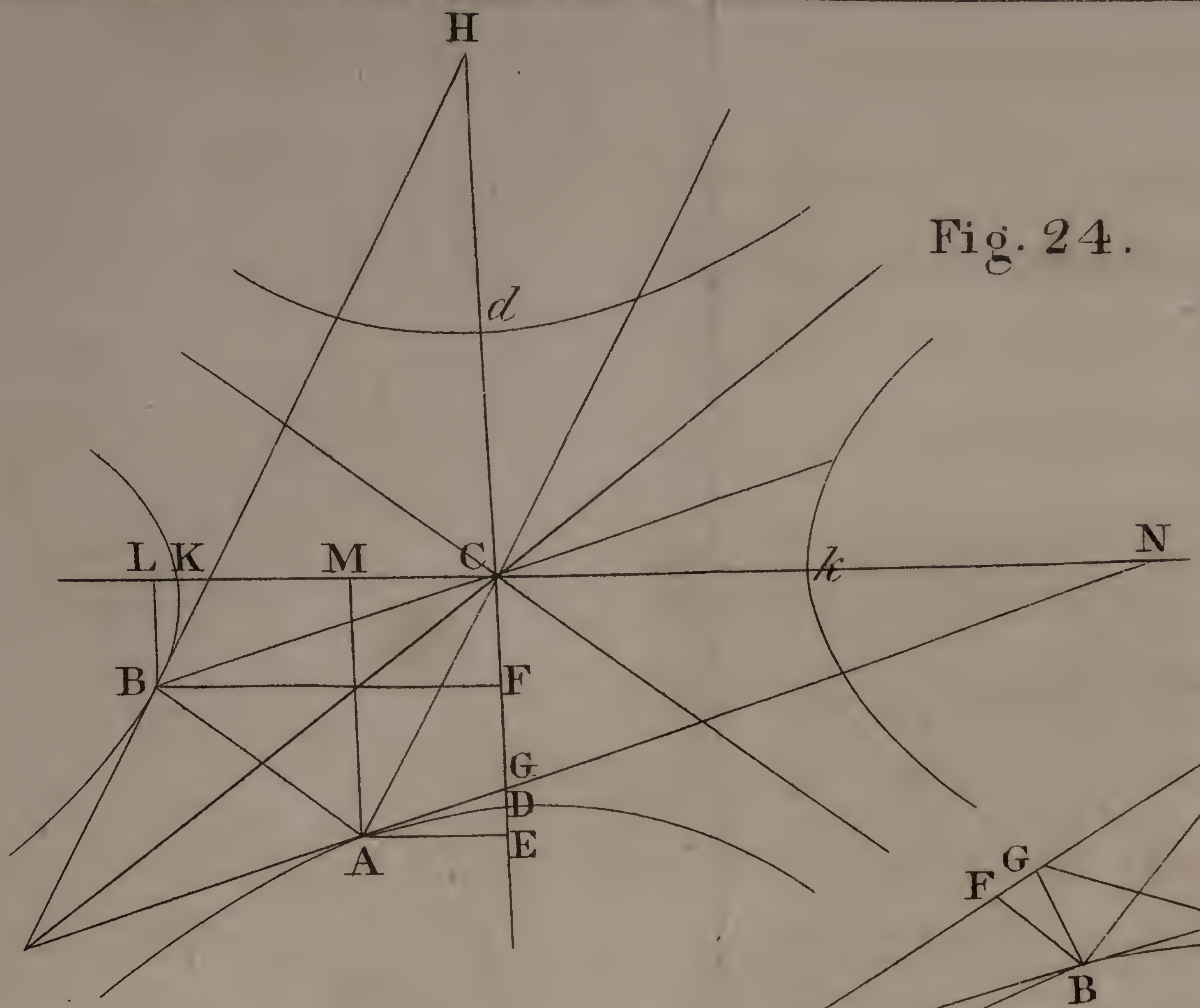


Fig. 24.

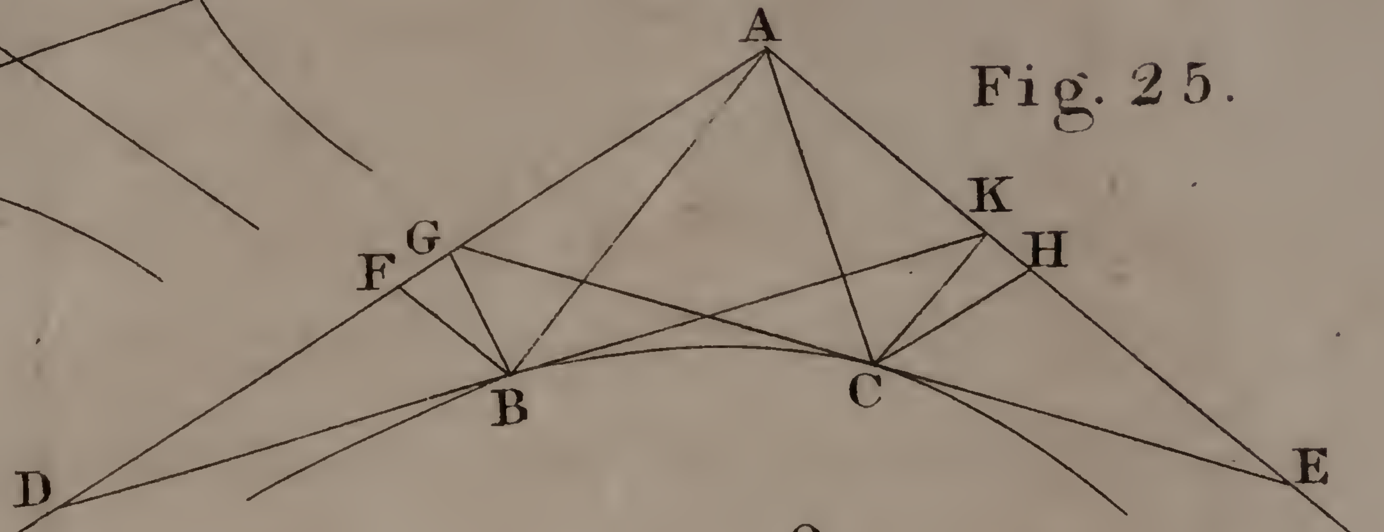


Fig. 25.

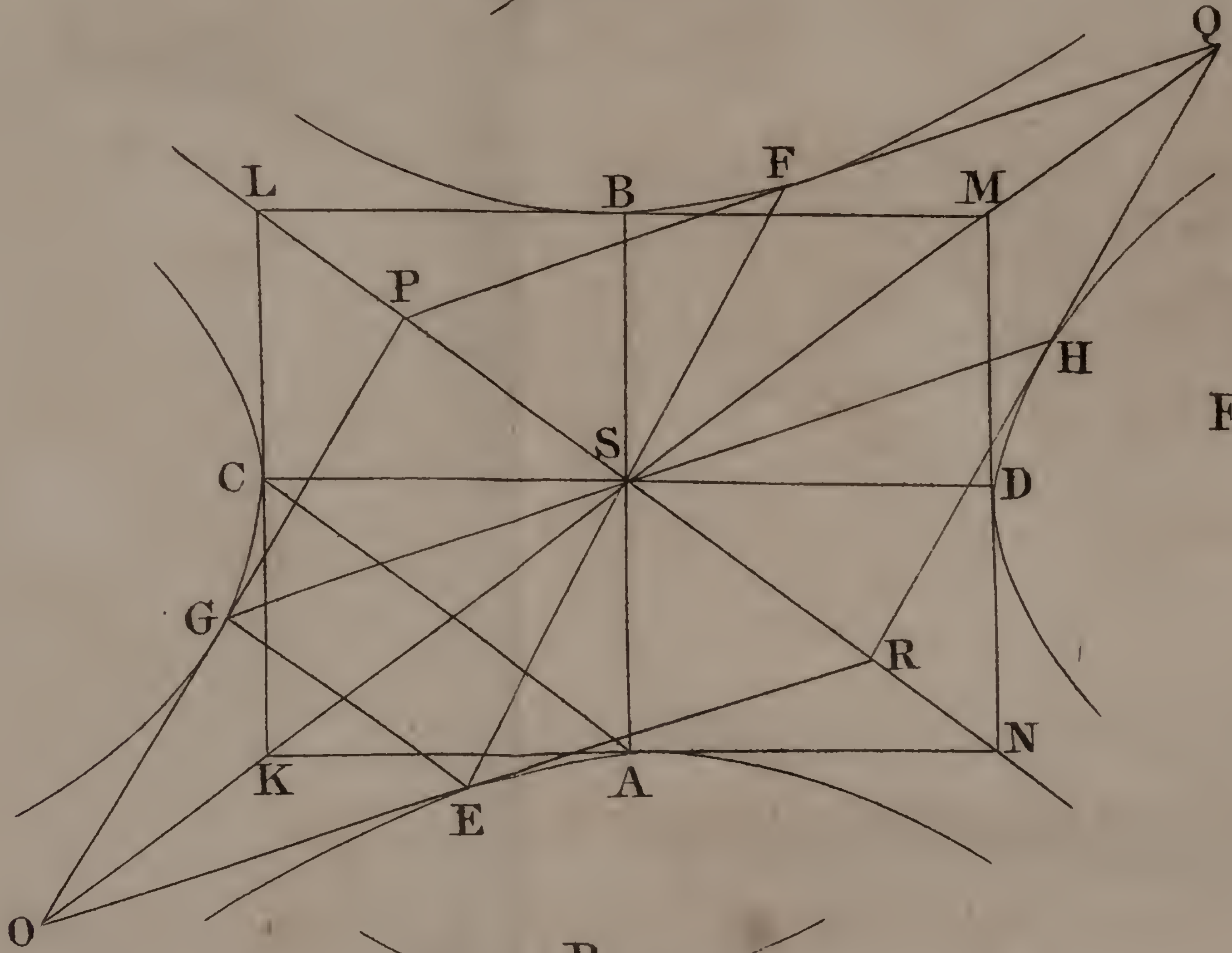


Fig. 26.

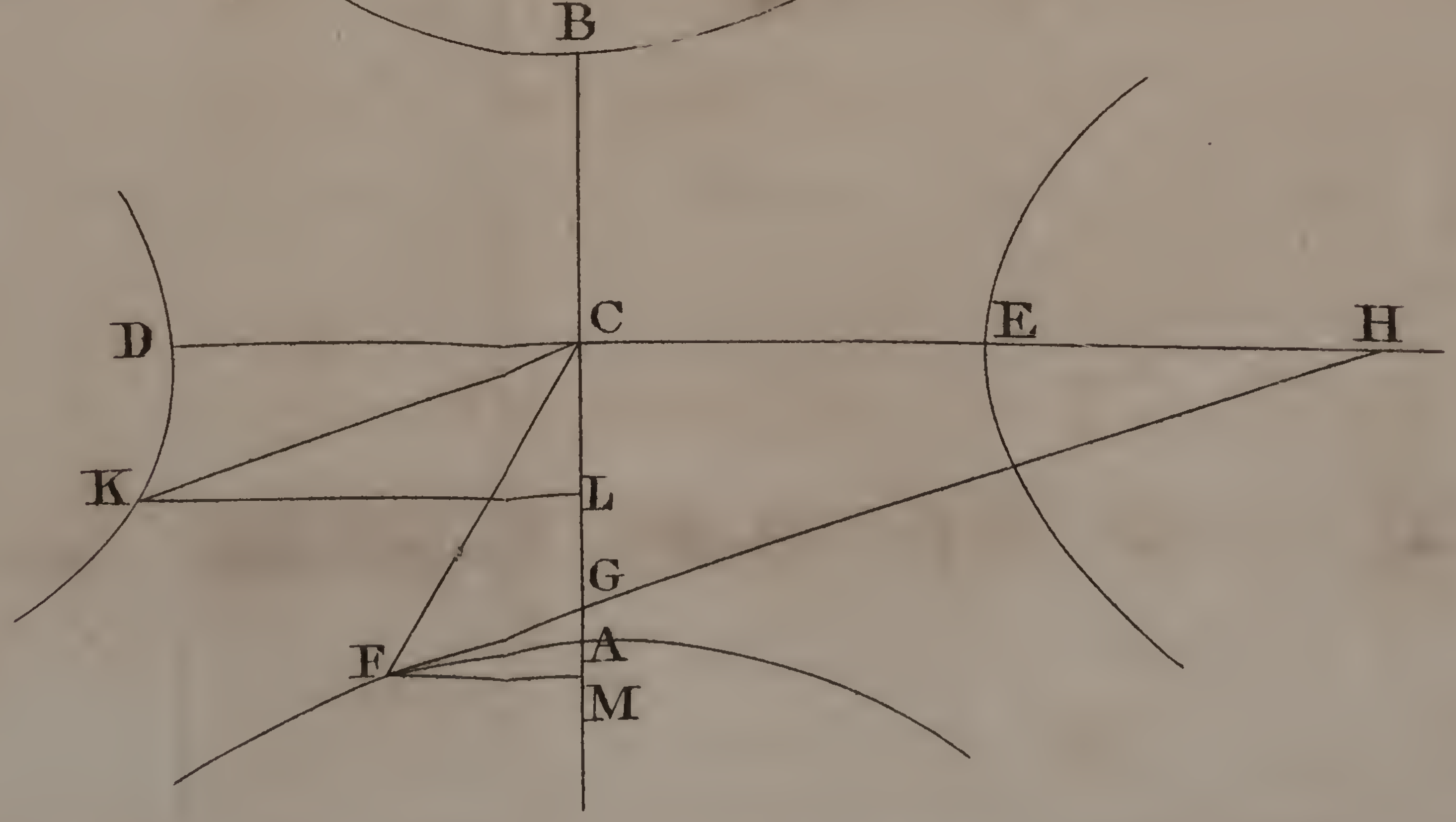


Fig. 27.

because of the parallels, GM is to MC, as GF to FH; consequently the rectangles GMC, GFH are similar: and because the triangles GMF, CLK are equiangular, GM is to GF, as CL to CK; therefore the rectangles GMC, GFH, and the squares of CL, CK, which are four similar and similarly situated rectilineal figures, described upon the four proportional straight lines GM, GF, CL, CK are likewise proportionals: but the rectangle GMC is equal to (35. 3.) the rectangle AMB, that is, to the (43. 3.) square of CL; therefore the rectangle GFH is equal to the square of CK.

PROP. XLVII. Theor

If from a point of a hyperbola a straight line be drawn ordinately applied to a transverse diameter, the rectangle contained by the segments of the diameter intercepted between its vertices and the ordinate, is to the square of the segment of the ordinate intercepted between the hyperbola and the diameter, as the diameter is to its *latus rectum:* but if a straight line be drawn ordinately applied to the second diameter of the transverse, the sum of the squares of half the second diameter, and of its segment between the ordinate and the centre, is to the square of the ordinate, as the second diameter is to its *latus rectum.*

Fig. 28. Let there be a transverse diameter AB, and DE the second diameter conjugate to it, and let AH be the *latus rectum* of AB, and from the point F in the hyperbola draw FG ordinately applied to AB; the rectangle AGB is to the square of FG, as AB to AH: but FK being drawn ordinately applied to DE; the sum of the squares of CD, CK is to the square of FK, as DE to its *latus rectum* L.

Case 1. Since AB, DE, AH are proportionals (def. 12.), AB is to AH, as the square of AB to the square of DE, that is (28. 3.), as the rectangle AGB to the square of FG.

Case 2. And since DE, AB, L are proportionals, DE is to L, as the square of DE is to the square of AB, that is (29. 3.), as the sum of the squares of CD, CK is to the square of KF.

PROP. XLVIII. Theor.

Fig. 23.

If from a point F in a hyperbola a straight line FG be ordinately applied to a transverse diameter AB, and from the vertex of that diameter a straight line AH be drawn perpendicular to AB, and equal to its *latus rectum;* the square of the ordinate is equal to the rectangle applied to the *latus rectum,* having for its breadth the abscissa between the ordinate and the vertex, but exceeding by a figure similar, and similarly situated, to that which is contained by the diameter and the *latus rectum.*

Join BH, and from the point G draw GM parallel to AH, and meeting BH in M, and and through the point M draw MN parallel to AB, and meeting AH in N; and complete the rectangles MNHO, BAHP.

Then because the rectangle AGB is to the square of FG, as AB to AH, that is, as GB to GM, that is, as the rectangle AGB to the rectangle AGM; therefore AGB is to the square of GF, as the same AGB to the rectangle AGM; consequently the square of GF is equal to the rectangle AGM, having the abscissa AG for its breadth, and applied to the *latus rectum* AH, and exceeding the rectangle HAGO, by the rectangle MNHO, similar to BAHP. From the square of the ordinate being thus equal to the *exceeding* rectangle, or that under the abscissa and a a line *greater* than the *latus rectum*, Apollonius called this curve line the *hyperbola*.

Cor. It is evident, that the square of GF would be equal to the rectangle AGM, though AH were not at right angles to AB.

PROP. XLIX. PROB.

Fig. 29

A straight line AB being given in position and magnitude, and a point F being given; to describe a hyperbola, of which AB may be the transverse axis, and which may pass through the point F; but the given point must be so situated, that a perpendicular drawn from it towards AB may fall upon AB produced.

Draw FG at right angles to AB, and find a straight line DE such,* that the square of AB may be to that of DE as the rectangle AGB to the square of FG: let AB, DE bisect each other at right angles; then, with AB, DE as axes, and making AB the transverse axis, describe a (37. 3.) hyperbola AF; this hyperbola will pass (2. cor. 28. 3.) through the point F.

---

* See note prop. 25. B. II.

PROP. L. PROB.

Fig. 29. A straight line DE being given in position and magnitude, and a point F being given; to describe a hyperbola, of which DE may be the second axis, and which may pass through F.

Bisect DE in C, and draw FH perpendicular to DE, and find a straight line AB such, * that the square of DE may be to that of AB, as the sum of the squares of CD, CH to the square of FH; let DE, AB bisect each other at right angles; then with the axes AB, DE, and making AB the transverse axis, describe (37. 3.) a hyperbola; this hyperbola will pass through the point F (2. cor. 29. 3.).

---

* Find a straight line X such, that its square may be equal to the sum of the squares of CD, CH (47. 1. Elem.); and to the three straight lines X, FH and DE find (12. 6. Elem.) a fourth proportional, which will be the transverse axis AB. For (22. 6. Elem.) the square of X, that is, the sum of the squares of CD, CH is to the square of FH, as the square of DE to the square of AB.

PROP. LI.

Of all transverse diameters in a hyperbola, the transverse axis is the least; and the angle contained by any other transverse diameter, and a tangent drawn through its vertex, is less than a right angle.

Fig. 29.

Let there be a hyperbola, CA the half of its transverse axis, and CF the half of any other transverse diameter; from F, the vertex of CF, draw FG perpendicular to the axis CA; therefore CF is greater than CG; and consequently much greater than CA: draw a straight line touching the hyperbola in the point F, and meeting the axis CA in K; and since the angle CFG is acute, CFK must be still more acute.

PROP. LII. Prob.

Fig. 29.

Of a hyperbola AF given in position, to find a diameter, the centre, the axes, the asymptotes, and the foci.

Draw two parallel straight lines, and let them be terminated both ways by the hyperbola; and the straight line which bisects them is a (3. cor. 31. 3.) diameter; and any other diameter may be found in the same manner; and the point where two diameters thus found meet each other is the (4. def. 3.) centre. But if two opposite hyperbolas be given in position, the point which bisects the diameter first found is the centre.

Take in the hyperbola any point F, and from the centre C draw CF, and with the centre C, and distance CF, describe a circle; if this circle meet the hyperbola no where but in the point F, CF is the least of the transverse diameters, and is, consequently, the transverse axis: but if the circle meet the hyperbola again in another point L, join FL, and let it be bisected in the point G; join also CG, and let it meet the hyperbola in A; CA is half the transverse axis: for since FG, GL are equal, FL is ordinately applied to the diameter CG; and consequently a straight line which is drawn through the vertex A parallel to FL touches the hyperbola (32. 3.); and the angle contained by this tangent, and the diameter CA, is a right angle: for the angle FGA is a right angle; therefore, by the preceding proposition, CA is the transverse axis.

Next, in order to find the second axis, draw through the centre C a straight line at right angles to CA, and in that straight line take CD, and let the square of CA have the same ratio to the square of CD, which the rectangle BGA has (CB being made equal to CA) to the square of GF; then CD will be the second axis, as is evident from prop. 7. of this book.

Lastly, having found the axes, find the asymptotes from def. 10.

But if two opposite hyperbolas be given in position, the asymptotes may be found more easily in this manner. Draw through the centre C any transverse diameter AB; draw likewise a straight line parallel to AB, and terminated in the hyperbolas in the points O, P; and to the straight line OP apply, on both sides, a rectangle equal to the square of CA, and deficient by a square; which is possible, since CA is less than the half of OP (4. cor. 15. 3.); and let Q, R be the points of application; CQ, CR, when joined, will be the asymptotes (3. cor. 15. 3.). The foci are found as in prop. 37.

PROP. LIII. Prob.

The asymptotes of a hyperbola being given in position, and a point in it being given; to find the axes of the hyperbola, and to describe it.

Fig. 30. Let AC, BC be the asymptotes given in position, and D be the given point. Suppose the problem solved, to wit, let FCE, GCH be the axes, the former of which, as it is within the angle ACB, in which the point D is, must be the transverse axis: draw through E a straight line parallel to GH, and let this parallel meet the asymptotes in the points K, L; consequently KL is equal (10. def. 3.) to GH, and is bisected in E: and since in the triangles KEC, LEC the bases KE, EL are equal, and the angles at E right angles; the angles ECK, ECL are equal: but the angle KCL is given; consequently its half KCE is given: and KC, and the point C, are given

in position; consequently CE is given (29. dat.) in position: through the given point D let DMN be drawn parallel to CE, and let it meet the asymptotes in M, N; DN is therefore given in (28. dat.) position, and the points M, N (25. dat.) are given; therefore DM, DN are given in magnitude (26. dat.); the rectangle MDN is consequently given in magnitude: but the square of CE is equal to this rectangle (1. cor. 15. 3.); therefore the square of CE is given in magnitude; and consequently CE is given (55. dat.) in magnitude; but, as formerly proved, it is also given in position; therefore the point E and the straight line KEL, are (27. 29. dat.) given in position; and consequently KEL is given in magnitude, because CA, CB are given in position: now GH is parallel and equal to KL; consequently GH is given in magnitude; but it is also given in position, because C is given, which bisects it; therefore the axes EF, GH are given in position and magnitude: and therefore the hyperbola may be described by prop. 37. of this book.

The composition is as follows. Let the angle ACB be bisected by the straight line

CE; and having drawn DMN parallel to CE, make the squares of CE, CF each of them equal to the rectangle MDN; and through E draw KEL perpendicular to CE, and meeting the straight lines AC, CB in the points K, L; and through C let GCH be drawn equal and parallel to KEL, so that it may be bisected in C: then, with the axes EF, GH, and making EF the transverse axis, describe a hyperbola; AC, BC will be its asymptotes, and it will pass through the point D. For since, by construction, KEL is equal and parallel to the second axis GH, and is bisected in E; therefore CK, CL are (def 10. 3.) the asymptotes, and the rectangle MDN is equal to the square of CE; consequently the point D is in the hyperbola (cor. 20. 3.).

And the asymptotes AC, BC being given, and a point D of a hyperbola, as many points of that hyperbola, or of the opposite hyperbola, as may be thought necessary, may be found, by drawing through D any number of straight lines ADB, D*ab*, meeting the asymptotes in A, B, and *a*, *b*; and taking BO, *bo* equal to AD, *a*D, in such a manner, that the two points D, O, and the two D, *o*, may be either both within or without the points A, B, and *a*, *b*: for then the points O, *o* will be (19. 3.) in the hyperbolas.

PROP. LIV. PROB.

Fig. 31.

Two straight lines ACB, DCE, which bisect each other in C, being given in position and magnitude; to describe two opposite hyperbolas, which may have AB for a transverse diameter, and DE for the second diameter conjugate to AB.

Suppose what is required done, and let CF, CG be the asymptotes; through A, the vertex of the transverse diameter, draw a straight line parallel to DE, and let it meet the asymptotes in F, G; therefore FG is (11. def. 3.) equal to DE, and is bisected in A: but DE is given in magnitude; therefore FG is given in magnitude; and of consequence its half AF is also given in magnitude: but AF is given (28. dat.) in position, since it is drawn through a given point A parallel to DE given in position; consequently the point F is (27. dat.) given: in like manner the point G is given; and the point C is given; therefore the asymptotes CF, CG are given in position, and the point A is given in the hyperbola: and therefore the hyperbola may be described, by the preceding proposition.

The composition is as follows. Through the vertex A of the transverse diameter, draw a straight line FAG equal and parallel to the second diameter DE, and so that it may be bisected in A; join CF, CG, and by the preceding prop. describe a hyperbola which may have for its asymptotes the straight lines CF, CG, and which may pass through the point A: and, in the same manner, by employing the point B, describe the opposite hyperbola; AB will be a transverse diameter in these hyperbolas, and DE the second diameter conjugate to it.

For since CF, CG are the asymptotes of the hyperbolas, and that through the point A, in the one of the hyperbolas, there is drawn a straight line FAG, which is bisected in A: FG touches (23. 3.) the hyperbola in A: and

DE is equal and parallel to FG, and is bisected in the centre C; therefore DE is the second, or conjugate diameter (11. def. and 30. 3.) to the transverse AB.

PROP. LV. Prob.

The diameter of a hyperbola being given in position and magnitude, and a straight line which is ordinately applied to that diameter from a given point of the hyperbola being also given in position; to describe the hyperbola.

Case 1. When the given diameter is a transverse diameter of the hyperbola.

Fig. 31.

Let AB be the given transverse diameter, to which a straight line HK, given in position, is ordinately applied from a given point H of the hyperbola; and let AB be bisected in C, and through C draw a straight line parallel to HK; and in that straight line take CD and CE equal to each other, so that the rectangle AKB may be to the square of HK, as the square of AC to the square of CD, or

CE; and, by the preceding proposition, describe two opposite hyperbolas, of which AB may be a transverse diameter, and DE the second diameter conjugate to AB; one of these hyperbolas will pass through the point H (2. cor. 28. of this book.).

Case 2. When the given diameter is a second diameter.

Let DE be the given second diameter, to which HL given in position, and drawn from H, a given point in the hyperbola, is ordinately applied; and let DE be bisected in C, and through C draw a straight line parallel to HL, and in that straight line take CA and CB equal to each other, so that the sum of the squares of LC, DC may be to the square of HL, as the square of DC to the square of AC, or CB; and, by the preceding proposition, describe two opposite hyperbolas, having AB for a transverse diameter, and CD for the second diameter conjugate to AB: of these hyperbolas, the one which lies on the same side of DE with the point H, will pass through the point H (2. cor. 29. 3.).

PROP. LVI. THEOR.

If a cone cut by a plane through the axis, be cut likewise by a second plane, meeting its base in the direction of a straight line perpendicular to the base of the triangle through the axis; and if the common section of the triangle through the axis, and the second plane, meet one of the sides of the triangle through the axis on the other side of the vertex of the cone; the line which is the common section of the second plane, and the conical surface, is a hyperbola, having for a transverse diameter the common section of the triangle through the axis and the second plane.

Let there be a cone, its vertex the point A, and base the circle BC; let it be cut by a plane through the axis, and let the triangle ABC be the section; let it be cut also by Fig. 32.

another plane, meeting its base in the direction of the straight line DE perpendicular to BC, the base of the triangle ABC; and let the section made in the surface of the cone be the line DFE; and let the straight line FG, the common section of the triangle through the axis, and the second plane, be produced, and meet one of the sides CA, of the triangle through the axis, in the point H, on the other side of the vertex A; the line DFE is a hyperbola, which has FG for one of its transverse diameters.

For in the section DFE take any point K, and through K to FG draw KL parallel to DE, and through L draw MN parallel to BC; the plane, therefore, which passes through KL, MN is (15. 11. Elem.) parallel to the plane through DE, BC, that is, to the base of the cone; and therefore the (23. 1.) plane through KL, MN is a circle, of which MN is a diameter: but KL is (10. 11. Elem.) perpendicular to MN, because DE is perpendicular to BC; therefore the rectangle MLN is (35. 3. Elem.) equal to the square of KL: and in like manner, the rectangle BGC is equal to the square of DG; the square of DG

PLATE XIV.

Fig. 28.

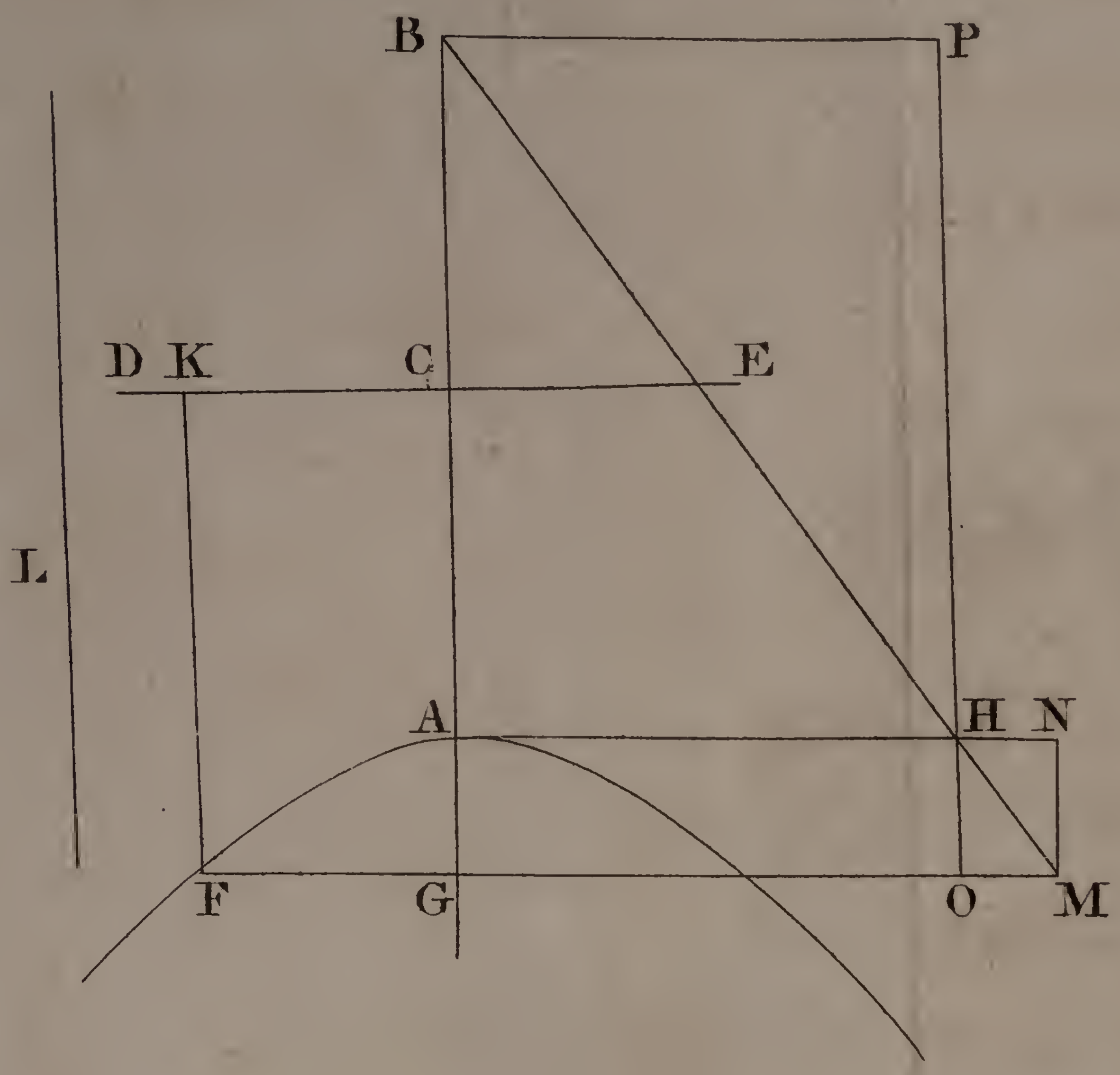
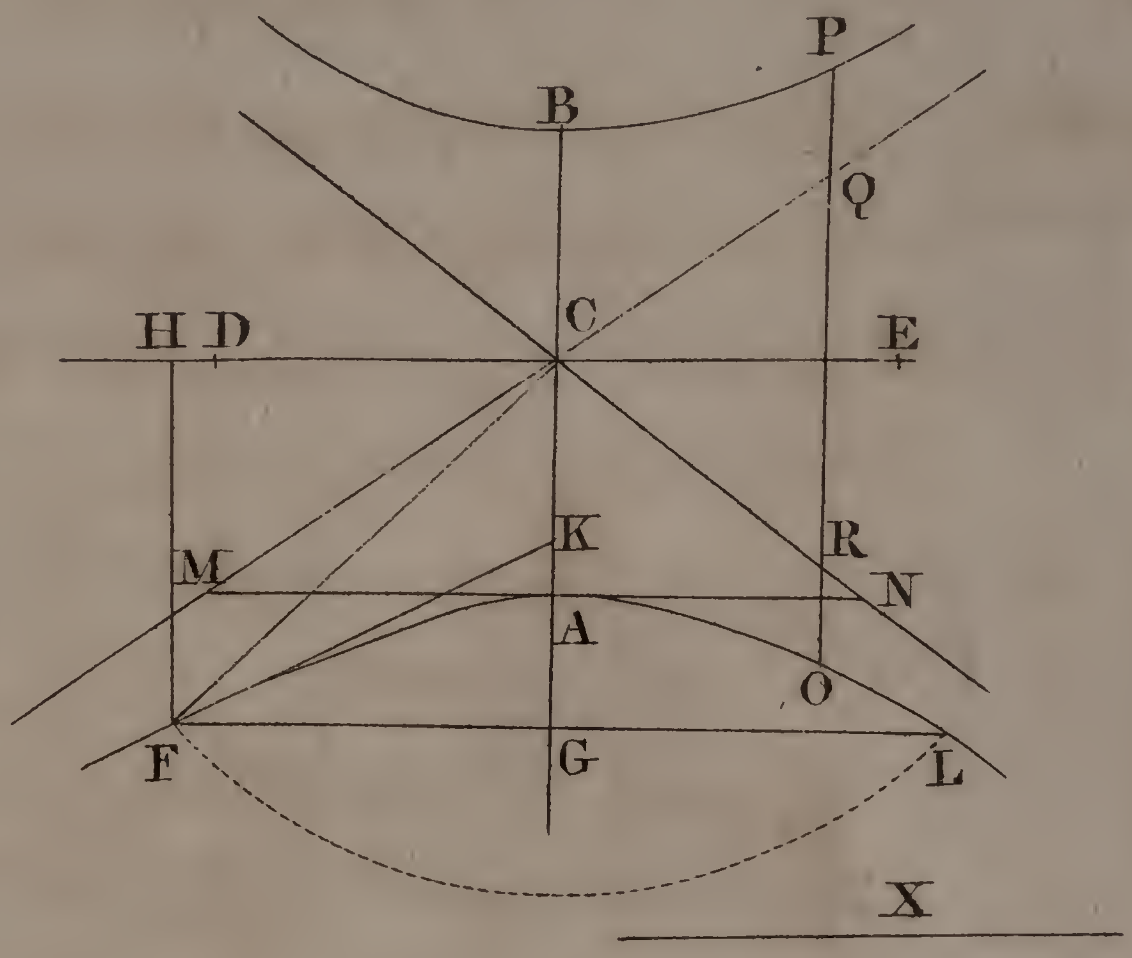


Fig. 29.



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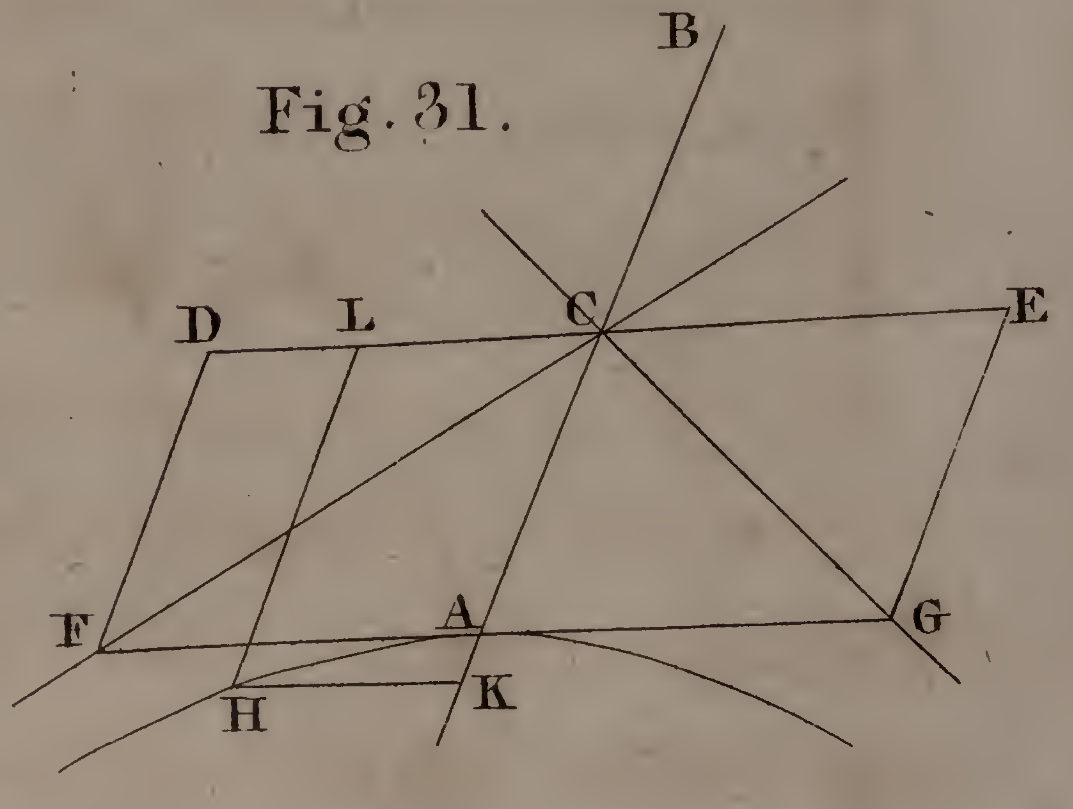
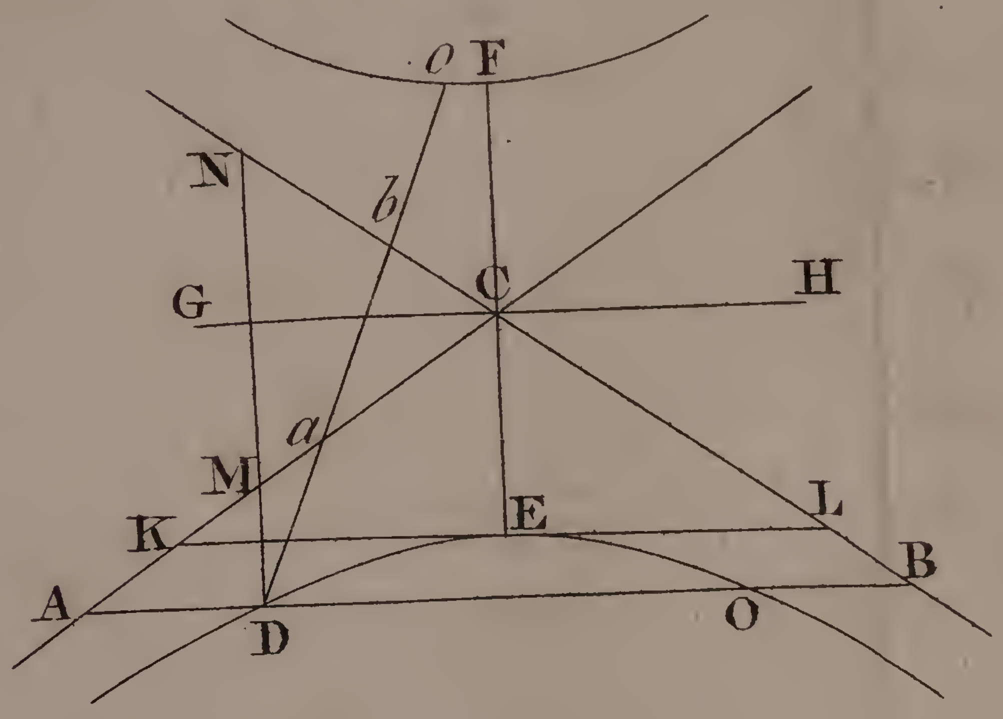


Fig. 30.



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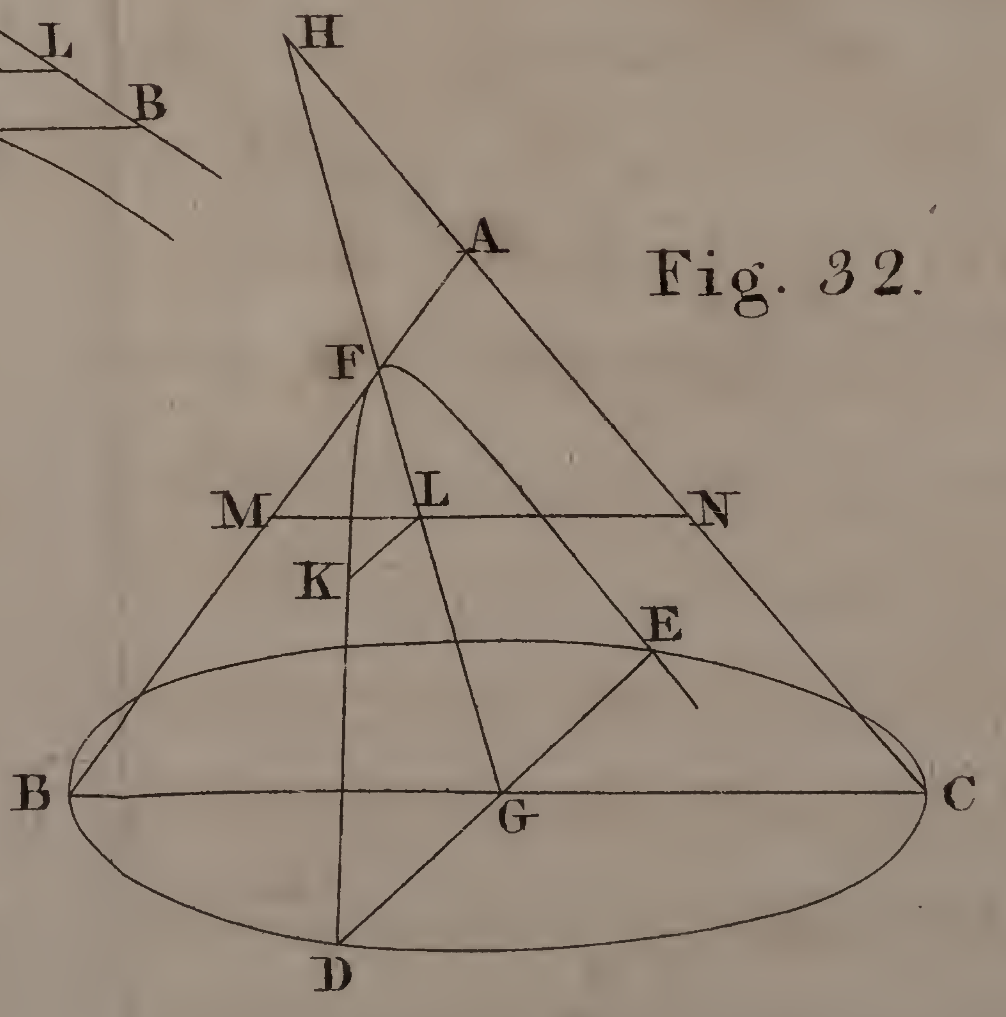

is therefore to the square of KL, as the rectangle BGC to the rectangle MLN: but BG is to ML, as FG to FL; and GC is to LN, as GH to LH; therefore the ratios compounded of these equal ratios are equal to one another; and therefore the rectangle BGC is to (23. 6. Elem.) the rectangle MLN, as the rectangle FGH to the rectangle FLH: hence, in like manner, the square of DG is to the square of KL, as the rectangle FGH to the rectangle FLH. Describe, therefore, a hyperbola (55. 3.), of which FH may be a transverse diameter, and in which DG may be ordinately applied to FH: and because, by construction, the point D is in this hyperbola, the point K is likewise in it (3. cor. 28. 3.). And the same thing may be proved with regard to all the points of the section DFE.

FINIS

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